On solvability of gas dynamics equations

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What is the "right" way of solving PDE?

However beautiful the strategy, you should occasionally look at the results...
Sir Winston Churchill
[1874-1965]

Euler system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

Energy conservation

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Constitutive relations

$$p = (\gamma - 1)\varrho e, \ \ e = c_v \vartheta, \ \ E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e$$

Entropy balance

$$\partial_t (\varrho s) + \mathrm{div}_x (s \mathbf{m}) = \boxed{\geq} 0$$

Classical (smooth) solutions

Initial data, boundary conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0,\ \vartheta(0,\cdot) = \vartheta_0$$

$$\mathbf{m}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

Local existence

Smooth initial data + smooth domain $\Rightarrow$ smooth (local) solutions in $[0, T_{\max})$, $T_{\max} < \infty$

Weak solutions

Mass conservation

$$\int\int \left(\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi\right)\mathrm{d}x\mathrm{d}t = 0$$

Momentum balance

$$\int\int \left(\mathbf{m}\cdot\boldsymbol{\varphi} + \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) : \nabla_x\boldsymbol{\varphi} + p\,\mathrm{div}_x\boldsymbol{\varphi}\right)\mathrm{d}x\mathrm{d}t = 0$$

Energy conservation

$$\int\int \left(E\partial_t\varphi + \left[(E+p)\,\frac{\mathbf{m}}{\varrho}\right]\cdot\nabla_x\varphi\right)\mathrm{d}x\mathrm{d}t = 0$$

Entropy balance

$$\int\int \left(\varrho s\partial_t\varphi + s\mathbf{m}\cdot\nabla_x\varphi\right)\mathrm{d}x\mathrm{d}t \le 0 \quad \varphi \ge 0$$

Well/ill posedness in the class of weak solutions

Ill posedness for bounded measurable data

There is a set of the data $\{\varrho_0, \vartheta_0\}$, dense in L^2, such that for any data from this set, there is $\mathbf{m}_0$ such that the Euler system admits infinitely many global–in–time weak solutions

Ill posedness for regular data

There exist (infinitely many) *Lipschitz* initial data $[\varrho_0, \mathbf{m}_0, \vartheta_0]$ such that the Euler system admits infinitely many global–in–time weak solutions

Driven system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = \varrho \nabla_x F$$

Energy conservation

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = \nabla_x F \cdot \mathbf{m}$$

Constitutive relations

$$p = (\gamma - 1)\varrho e, \ \ e = c_v \vartheta, \ \ E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e$$

Entropy balance

$$\partial_t (\varrho s) + \mathrm{div}_x (s\mathbf{m}) \geq 0$$

Stochastic perturbation

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = -\frac{1}{2} \mathbf{m} \circ \mathrm{d} W$$

Energy conservation

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = -E \circ \mathrm{d} W$$

Constitutive relations

$$p = (\gamma - 1) \varrho e, \ \ e = c_v \vartheta, \ \ E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e$$

Entropy balance

$$\partial_t (\varrho s) + \mathrm{div}_x (s \mathbf{m}) \geq -c_v \varrho \circ \mathrm{d} W$$