Stationary solutions to stochastically driven compressible Navier–Stokes system

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Driven Navier-Stokes system

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\mathrm{d}t + \boxed{\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Stochastic forcing

$$\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(x, \varrho, \mathbf{u})\mathrm{d}W_k$$

Iconic examples

$$\mathbf{G}_k = \mathbf{f}_k(x), \ \mathbf{G}_k = \mathbf{u} d_k(x) - \text{“stochastic damping”}$$

Additive vs. multiplicative noise

Additive noise

$$\varrho \sum_{k=1}^{\infty} b_k(x)\ \mathrm{d}W_k$$

$$\mathrm{d}(\varrho \mathbf{u}) \cdots = \varrho \sum_{k=1}^{\infty} b_k(x)\ \mathrm{d}W_k = \mathrm{d}\left(\varrho \sum_{k=1}^{\infty} b_k(x)\ \mathrm{d}W_k\right) - \mathrm{d}\varrho \sum_{k=1}^{\infty} b_k(x) W_k$$

$$= \mathrm{d}\left(\varrho \sum_{k=1}^{\infty} b_k(x)\ \mathrm{d}W_k\right) + \mathrm{div}_x(\varrho \mathbf{u}) \sum_{k=1}^{\infty} b_k(x) W_k$$

General multiplicative noise

$$\varrho \mathbf{G}(x, \varrho, \mathbf{u}) \mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(x, \varrho, \mathbf{u}) \mathrm{d}W_k$$

Initial and boundary conditions

(Random) initial data

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0$$

Spatial domain

$$Q \subset R^N, \text{ or "flat" torus } Q = \mathcal{T}^N = \left([0,1]|_{\{0,1\}}\right)^N,\ N = (1), 2, 3$$

$$\mathbf{u}\cdot\mathbf{n}|_{\partial Q} = 0 \text{ impermeability}$$

$$\mathbf{u}\times\mathbf{n}|_{\partial Q} = 0 \text{ no-slip}$$

$$[\mathbb{S}\cdot\mathbf{n}]\times\mathbf{n}|_{\partial Q} = 0 \text{ complete slip}$$

Weak (PDE) formulation

Field equations

$$\left[\int_Q \varrho \phi \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_Q \varrho \mathbf{u} \cdot \nabla_x \phi \, \mathrm{d}x\mathrm{d}t,$$

$$\left[\int_Q \varrho \mathbf{u} \cdot \phi \, \mathrm{d}x \right]_{t=0}^{t=\tau} - \int_0^\tau \int_Q \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \phi + p(\varrho) \mathrm{div}_x \phi \, \mathrm{d}x\mathrm{d}t$$

$$= - \int_0^\tau \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \phi \, \mathrm{d}x\mathrm{d}t + \boxed{\int_0^\tau \left(\int_Q \varrho \mathbf{G} \cdot \phi \, \mathrm{d}x \right) \mathrm{d}W}$$

$\phi = \phi(x) -$ a smooth test function

Stochastic integral (Itô's formulation)

$$\int_0^\tau \left(\int_Q \varrho \mathbf{G} \cdot \phi \, \mathrm{d}x \right) \mathrm{d}W = \sum_{k=1}^\infty \int_0^\tau \left(\int_Q \varrho \mathbf{G}_k \cdot \phi \, \mathrm{d}x \right) \mathrm{d}\beta_k$$

Admissibility

Energy inequality

$$-\int_0^T \partial_t\psi \int_Q \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right]\,\mathrm{d}x\mathrm{d}t + \int_0^T \psi \int_Q \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,\mathrm{d}x\mathrm{d}t$$

$$\leq \psi(0)\int_Q \left[\frac{|(\varrho\mathbf{u})_0|^2}{2\varrho_0} + P(\varrho_0)\right]\,\mathrm{d}x$$

$$+\frac{1}{2}\int_0^T \psi\left(\int_Q \sum_{k\geq 1}\varrho|\mathbf{G}_k(x,\varrho,\mathbf{u})|^2\,\mathrm{d}x\right)\mathrm{d}t + \int_0^T \psi\mathrm{d}M_E$$

$$\psi \geq 0,\ \psi(T) = 0,\ P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z$$

Strong vs. martingale solutions

Strong solutions

- the functions ϱ, $\mathbf{u}$ are differentiable a.s., the equations are satisfied in the classical sense
- the probability space uniquely determined

Martingale solutions

- solutions defined on a different, typically, the standard probability space
- the white noise as well as the initial data coincide with the originals *in law*

Existence theory

Local existence of strong solutions [Kim [2011]], [Breit, EF, Hofmanová [2017]]

If the initial data are smooth, then the problem admits local-in-time smooth solutions. Solutions exist up to a (maximal) positive *stopping time*. The life-span is a random variable.

Global existence for the Navier–Stokes system [Breit, Hofmanová [2015]]

The Navier–Stokes system admits global–in–time martingale solutions for

$$p(\varrho) \approx \varrho^{\gamma},\ \gamma > \frac{N}{2}$$

Relative energy inequality

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_Q \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)\right] \mathrm{d}x$$

Relative energy inequality

$$-\int_0^T \partial_t\psi \; \mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \; \mathrm{d}t$$

$$+\int_0^T \psi \int_Q \mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \; \mathrm{d}x\mathrm{d}t$$

$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi \mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \mathrm{d}t$$

Test functions

$$\mathrm{d}r = D_t^d r \; \mathrm{d}t + \mathbb{D}_t^s r \; \mathrm{d}W, \;\; \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \; \mathrm{d}t + \mathbb{D}_t^s \mathbf{U} \; \mathrm{d}W$$

Remainder

Remainder term

$$\mathcal{R}\left(\varrho,\mathbf{u}\middle| r,\mathbf{U}\right) = \int_Q \varrho\Big(D_t^d\mathbf{U} + \mathbf{u}\cdot\nabla_x\mathbf{U}\Big)(\mathbf{U}-\mathbf{u})\ \mathrm{d}x$$

$$+\int_Q \left((r-\varrho)P''(r)D_t^d r + \nabla_x P'(r)(r\mathbf{U}-\varrho\mathbf{u})\right)\ \mathrm{d}x$$

$$-\int_Q \mathrm{div}_x\mathbf{U}(p(\varrho)-p(r))\ \mathrm{d}x$$

$$+\frac{1}{2}\sum_{k\geq 1}\int_Q \varrho\Big|\mathbf{G}_k(\varrho,\varrho\mathbf{u}) - [\mathbb{D}_t^s\mathbf{U}]_k\Big|^2\ \mathrm{d}x$$

$$+\frac{1}{2}\sum_{k\geq 1}\int_Q \varrho P'''(r)|[\mathbb{D}_t^s r]_k|^2\ \mathrm{d}x + \frac{1}{2}\sum_{k\geq 1}\int_Q p''(r)|[\mathbb{D}_t^s r]_k|^2\ \mathrm{d}x$$

$$+\int_Q \mathbb{S}(\nabla_x\mathbf{U}) : (\nabla_x\mathbf{U}-\nabla_x\mathbf{u})\ \mathrm{d}x$$

Weak–strong uniqueness

Weak–strong uniqueness [Breit, EF, Hofmanová [2016]]

Pathwise uniqueness.
A weak and strong solutions defined on the same probability space and emanating from the same initial data coincide as long as the latter exists
Uniqueness in law.
If a weak and strong solution are defined on a different probability space, then their *laws* are the same provided the laws of the initial data are the same

Stochastic stationary solutions

Complete trajectories

$$\varrho, \mathbf{u} \in L^2_{\rm loc}(0, T; L^\gamma \times W^{1,2}(Q; R^N))$$

Stationarity

The law of the solution is time shift invariant

$$\text{probability}\,\{(\varrho, \mathbf{u}) \in B\} = \text{probability}\,\{(\varrho(t+\tau), \mathbf{u}(t+\tau)) \in B\}$$

for any $\tau \geq 0$ and any

$$B - \text{ a Borel subset of } L^2_{\rm loc}(0, T; L^\gamma \times W^{1,2}(Q; R^N))$$

Stationary solutions to the Navier–Stokes system

Basic hypotheses

- $$|\mathbf{G}_k| + |\nabla \mathbf{G}_k| \approx \alpha_k, \ \sum_{k>0} \alpha_k^2 < \infty$$
- $$p(\varrho) \approx \varrho^\gamma, \ \gamma > \frac{N}{2}$$
- complete slip/no slip boundary conditions

Stationary solutions [Breit, EF, Hofmanová, Maslowski] [2017]

For a given (deterministic) mass

$$M = \int_Q \varrho \, \mathrm{d}x > 0$$

the Navier–Stokes system admits a stationary martingale solution.

Method of the proof

Finite–dimensional approximatio

Use the Krylov–Bogolyubov theory on the approximate system

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u}) = \varepsilon \Delta_x \varrho + M \left(\int_Q \varrho \, \mathrm{d}x \right)$$

\+ Galerkin approximation for the momentum equation

Uniform bounds

Uniform bounds based on deterministic estimates + Itô's chain rule

Stochastic compactness method

Skorokhod–Prokhorov theorem (works on Polish spaces), here we have weak topology

Complete system – more physics?

Complete system

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho, \vartheta)\mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\mathrm{d}t + \boxed{\varrho \mathbf{G}(x, \varrho, \mathbf{u})\mathrm{d}W}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u}\mathbb{I}$$

Internal energy balance

$$\mathrm{d}\varrho e(\varrho, \vartheta) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u})\mathrm{d}t + \mathrm{div}_x \mathbf{q}\mathrm{d}t = \mathbb{S}(\nabla_x \mathbf{u}) : \mathbf{u}\mathrm{d}t - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}\mathrm{d}t$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Gibbs' relation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$