Convergence of graphons and the weak* topology

Martin Doležal
(joint work with J. Grebík, J. Hladký, I. Rocha, V. Rozhoň)

Institute of Mathematics of the Czech Academy of Sciences

Summer Symposium in Real Analysis
Санкт-Петербург
June 9-15, 2018

Motivation

How to understand very large networks?

Motivation

How to understand very large networks? (E.g. internet, social networks, neurons in the human brain etc.)

Motivation

How to understand very large networks? (E.g. internet, social networks, neurons in the human brain etc.)

A graph is a pair $G = (V, E)$ where V is a finite set and E is a set of 2-element subsets of V.

Motivation

How to understand very large networks? (E.g. internet, social networks, neurons in the human brain etc.)

A graph is a pair $G = (V, E)$ where V is a finite set and E is a set of 2-element subsets of V.

But huge networks are never completely known, often not even well defined.

Introduction

Two types of large networks:

Introduction

Two types of large networks:

- sparse networks (each vertex has a bounded number of neighbors)

Introduction

Two types of large networks:

- sparse networks (each vertex has a bounded number of neighbors)
- dense networks (each node is a neighbor of a positive percent of other vertices)

Introduction

Two types of large networks:

- sparse networks (each vertex has a bounded number of neighbors)
- dense networks (each node is a neighbor of a positive percent of other vertices)

We are interested in the approximation of dense networks by certain 'analytic' objects.

Introduction

Two types of large networks:

- sparse networks (each vertex has a bounded number of neighbors)
- dense networks (each node is a neighbor of a positive percent of other vertices)

We are interested in the approximation of dense networks by certain 'analytic' objects.

Our plan: Let (G_n) be a sequence of graphs whose number of vertices tends to infinity.

Introduction

Two types of large networks:

- sparse networks (each vertex has a bounded number of neighbors)
- dense networks (each node is a neighbor of a positive percent of other vertices)

We are interested in the approximation of dense networks by certain 'analytic' objects.

Our plan: Let (G_n) be a sequence of graphs whose number of vertices tends to infinity. When is such a sequence convergent?

Introduction

Two types of large networks:

- sparse networks (each vertex has a bounded number of neighbors)
- dense networks (each node is a neighbor of a positive percent of other vertices)

We are interested in the approximation of dense networks by certain 'analytic' objects.

Our plan: Let (G_n) be a sequence of graphs whose number of vertices tends to infinity. When is such a sequence convergent? What is the limit object?

Graphons and cut distance

A graphon is a symmetric Lebesgue measurable function $W\colon [0,1]^2 \to [0,1]$.

Graphons and cut distance

A graphon is a symmetric Lebesgue measurable function $W\colon [0,1]^2 \to [0,1]$.

How do we measure the distance of two graphons?

Graphons and cut distance

A graphon is a symmetric Lebesgue measurable function $W\colon [0,1]^2 \to [0,1]$.

How do we measure the distance of two graphons?

For every measurable function $V\colon [0,1]^2 \to [-1,1]$ we define the cut norm of V by

$$\|V\|_\square := \sup_{S,T\subseteq[0,1]} \int_{S\times T} V(x,y)$$

where the supremum ranges over all measurable sets $S, T \subseteq [0,1]$.

Graphons and cut distance

A graphon is a symmetric Lebesgue measurable function $W\colon [0,1]^2 \to [0,1]$.

How do we measure the distance of two graphons?

For every measurable function $V\colon [0,1]^2 \to [-1,1]$ we define the cut norm of V by

$$\|V\|_\square := \sup_{S,T\subseteq[0,1]} \int_{S\times T} V(x,y)$$

where the supremum ranges over all measurable sets $S, T \subseteq [0,1]$.

For graphons U, W we define the cut distance of U and W by

$$\delta_\square(U,W) := \inf_{\varphi\colon [0,1]\to[0,1]} \|U^\varphi - W\|_\square$$

where the infimum ranges over all invertible measure preserving maps $\varphi\colon [0,1] \to [0,1]$ and U^φ is defined by

$$U^\varphi(x,y) = U(\varphi(x),\varphi(y)).$$

Graphons and cut distance

The cut-distance $\delta_\square$ is a pseudometric.

Graphons and cut distance

The cut-distance $\delta_\square$ is a pseudometric. If we identify two graphons whenever their cut distance is 0 then the space of all (equivalence classes of) graphons is a metric space.

Graphons and cut distance

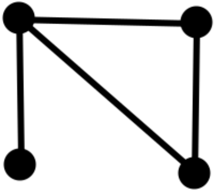
The cut-distance $\delta_\square$ is a pseudometric. If we identify two graphons whenever their cut distance is 0 then the space of all (equivalence classes of) graphons is a metric space.

Every graph can be represented by a graphon:

Graphons and cut distance

The cut-distance $\delta_\square$ is a pseudometric. If we identify two graphons whenever their cut distance is 0 then the space of all (equivalence classes of) graphons is a metric space.

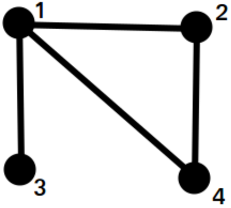
Every graph can be represented by a graphon:



Graphons and cut distance

The cut-distance $\delta_\square$ is a pseudometric. If we identify two graphons whenever their cut distance is 0 then the space of all (equivalence classes of) graphons is a metric space.

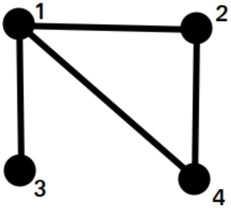
Every graph can be represented by a graphon:

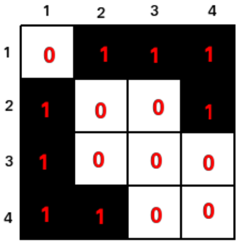


Graphons and cut distance

The cut-distance $\delta_\square$ is a pseudometric. If we identify two graphons whenever their cut distance is 0 then the space of all (equivalence classes of) graphons is a metric space.

Every graph can be represented by a graphon:



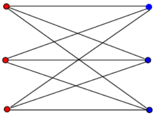


Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

Basic example

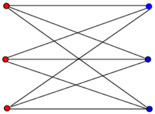
$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n



$K_{3,3}$

Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

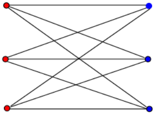


$K_{3,3}$

There are many representations of $K_{n,n}$.

Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n

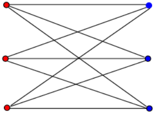


$K_{3,3}$

There are many representations of $K_{n,n}$. Here is one of them (for large n):

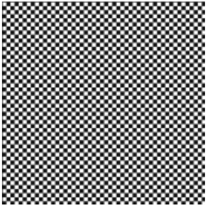
Basic example

$K_{n,n}$ ..... the complete bipartite graph with both partitions of size n



$K_{3,3}$

There are many representations of $K_{n,n}$. Here is one of them (for large n):



Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ *is compact.*

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ *is compact.*

Known proofs:

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:
  Szemerédi's regularity lemma

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:
  Szemerédi's regularity lemma
- Diaconis & Janson and (independently) Austin, 2008:

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:
  Szemerédi's regularity lemma
- Diaconis & Janson and (independently) Austin, 2008:
  Aldous-Hoover theorem on exchangeable arrays (1981)

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:
  Szemerédi's regularity lemma
- Diaconis & Janson and (independently) Austin, 2008:
  Aldous-Hoover theorem on exchangeable arrays (1981)
- Elek & Szegedy, 2012:

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:
  Szemerédi's regularity lemma
- Diaconis & Janson and (independently) Austin, 2008:
  Aldous-Hoover theorem on exchangeable arrays (1981)
- Elek & Szegedy, 2012:
  ultraproducts

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:
  Szemerédi's regularity lemma
- Diaconis & Janson and (independently) Austin, 2008:
  Aldous-Hoover theorem on exchangeable arrays (1981)
- Elek & Szegedy, 2012:
  ultraproducts
- Our proof:

Compactness of the cut-distance

Theorem (Lovász & Szegedy, 2006)

The space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$ is compact.

Known proofs:

- Lovász & Szegedy, 2006:
  Szemerédi's regularity lemma
- Diaconis & Janson and (independently) Austin, 2008:
  Aldous-Hoover theorem on exchangeable arrays (1981)
- Elek & Szegedy, 2012:
  ultraproducts
- Our proof:
  weak* topology of $L^\infty([0,1]^2)$

The weak* topology

Every graphon belongs to the unit ball of the dual Banach space $L^\infty([0,1]^2)$.

The weak* topology

Every graphon belongs to the unit ball of the dual Banach space $L^{\infty}([0,1]^2)$.

We consider the restriction of the weak* topology to the space of all graphons.

The weak* topology

Every graphon belongs to the unit ball of the dual Banach space $L^\infty([0,1]^2)$.

We consider the restriction of the weak* topology to the space of all graphons.

Then we have that...

The weak* topology

Every graphon belongs to the unit ball of the dual Banach space $L^\infty([0,1]^2)$.

We consider the restriction of the weak* topology to the space of all graphons.

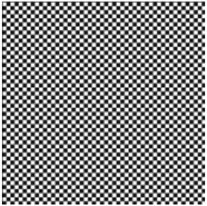
Then we have that...

...a sequence $(W_n)_n$ of graphons weak* converges to a graphon W iff for every measurable set $S \subseteq [0,1]$ it holds

$$\lim_{n\to\infty} \int_{S\times S} W_n(x,y) = \int_{S\times S} W(x,y).$$

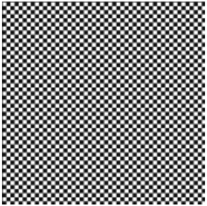
Basic example again

A representation of $K_{n,n}$ (for large n):



Basic example again

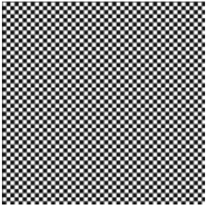
A representation of $K_{n,n}$ (for large n):



When $n \to \infty$ then

Basic example again

A representation of $K_{n,n}$ (for large n):

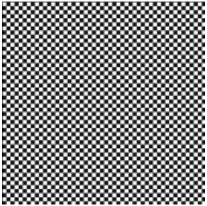


When $n \to \infty$ then

- these graphons weak* converge to the constant graphon $C_{\frac{1}{2}} \equiv \frac{1}{2}$

Basic example again

A representation of $K_{n,n}$ (for large n):

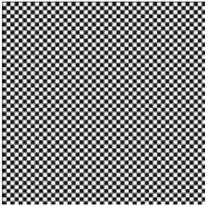


When $n \to \infty$ then

- these graphons weak* converge to the constant graphon $C_{\frac{1}{2}} \equiv \frac{1}{2}$
- all these graphons belong to the same equivalence class

Basic example again

A representation of $K_{n,n}$ (for large n):



When $n \to \infty$ then

- these graphons weak* converge to the constant graphon $C_{\frac{1}{2}} \equiv \frac{1}{2}$
- all these graphons belong to the same equivalence class
- therefore these graphons do not converge to $C_{\frac{1}{2}}$ in the cut distance

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons.

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find an accumulation point W of $(W_n)_n$ in the cut distance $\delta_\square$.

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find an accumulation point W of $(W_n)_n$ in the cut distance $\delta_\square$.

For a graphon W and an invertible measure preserving map $\varphi\colon [0,1] \to [0,1]$ we define the graphon W^φ by

$$W^\varphi(x,y) = W(\varphi(x), \varphi(y)).$$

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find an accumulation point W of $(W_n)_n$ in the cut distance $\delta_\square$.

For a graphon W and an invertible measure preserving map $\varphi\colon [0,1] \to [0,1]$ we define the graphon W^φ by

$$W^\varphi(x,y) = W(\varphi(x), \varphi(y)).$$

We define

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find an accumulation point W of $(W_n)_n$ in the cut distance $\delta_\square$.

For a graphon W and an invertible measure preserving map $\varphi\colon [0,1] \to [0,1]$ we define the graphon W^φ by

$$W^\varphi(x,y) = W(\varphi(x), \varphi(y)).$$

We define

$$\mathrm{LIM}_{w*}((W_n)_n) := \{W\colon \text{ there are invertible measure preserving maps } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* limit of } (W_n^{\varphi_n})_n\}$$

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find an accumulation point W of $(W_n)_n$ in the cut distance $\delta_\square$.

For a graphon W and an invertible measure preserving map $\varphi\colon [0,1] \to [0,1]$ we define the graphon W^φ by

$$W^\varphi(x,y) = W(\varphi(x), \varphi(y)).$$

We define

$$\mathrm{LIM}_{w*}((W_n)_n) := \{W\colon \text{ there are invertible measure preserving maps } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* limit of } (W_n^{\varphi_n})_n\}$$

and

Our proof of compactness

Let $(W_n)_n$ be a sequence of graphons. We need to find an accumulation point W of $(W_n)_n$ in the cut distance $\delta_\square$.

For a graphon W and an invertible measure preserving map $\varphi\colon [0,1] \to [0,1]$ we define the graphon W^φ by

$$W^\varphi(x,y) = W(\varphi(x), \varphi(y)).$$

We define

$$\mathrm{LIM}_{w*}((W_n)_n) := \{W\colon \text{ there are invertible measure preserving maps } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* limit of } (W_n^{\varphi_n})_n\}$$

and

$$\mathrm{ACC}_{w*}((W_n)_n) := \{W\colon \text{ there are invertible measure preserving maps } \varphi_n\colon [0,1] \to [0,1] \text{ such that } W \text{ is a weak* accumulation point of } (W_n^{\varphi_n})_n\}.$$

Our proof of compactness

We want to take the 'most structured' element of either $\mathrm{LIM}_{w*}((W_n)_n)$ or $\mathrm{ACC}_{w*}((W_n)_n)$ and prove that it is an accumulation point of $(W_n)_n$ in the cut distance $\delta_\square$.

Our proof of compactness

We want to take the 'most structured' element of either $\mathrm{LIM}_{w*}((W_n)_n)$ or $\mathrm{ACC}_{w*}((W_n)_n)$ and prove that it is an accumulation point of $(W_n)_n$ in the cut distance $\delta_\square$.

Problems:

Our proof of compactness

We want to take the 'most structured' element of either $\mathrm{LIM}_{w*}((W_n)_n)$ or $\mathrm{ACC}_{w*}((W_n)_n)$ and prove that it is an accumulation point of $(W_n)_n$ in the cut distance $\delta_\square$.

Problems:

- the set $\mathrm{LIM}_{w*}((W_n)_n)$ may be empty

Our proof of compactness

We want to take the 'most structured' element of either $\mathrm{LIM}_{w*}((W_n)_n)$ or $\mathrm{ACC}_{w*}((W_n)_n)$ and prove that it is an accumulation point of $(W_n)_n$ in the cut distance $\delta_\square$.

Problems:

- the set $\mathrm{LIM}_{w*}((W_n)_n)$ may be empty
- the set $\mathrm{ACC}_{w*}((W_n)_n)$ is nonempty (by Banach-Alaoglu theorem)

Our proof of compactness

We want to take the 'most structured' element of either $\mathrm{LIM}_{w*}((W_n)_n)$ or $\mathrm{ACC}_{w*}((W_n)_n)$ and prove that it is an accumulation point of $(W_n)_n$ in the cut distance $\delta_\square$.

Problems:

- the set $\mathrm{LIM}_{w*}((W_n)_n)$ may be empty
- the set $\mathrm{ACC}_{w*}((W_n)_n)$ is nonempty (by Banach-Alaoglu theorem) but the 'most structured' element of $\mathrm{ACC}_{w*}((W_n)_n)$ may not exist

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ *of graphons there is a subsequence* $(W_{n_k})_k$ *of* $(W_n)_n$ *such that*

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ *of graphons there is a subsequence* $(W_{n_k})_k$ *of* $(W_n)_n$ *such that*

$$\mathsf{ACC}_{w*}((W_{n_k})_k) = \mathsf{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_k)_k$ *of graphons the following conditions are equivalent:*

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ of graphons there is a subsequence $(W_{n_k})_k$ of $(W_n)_n$ such that

$$\mathsf{ACC}_{w*}((W_{n_k})_k) = \mathsf{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_k)_k$ of graphons the following conditions are equivalent:

- $\mathsf{ACC}_{w*}((W_k)_k) = \mathsf{LIM}_{w*}((W_k)_k)$,

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ *of graphons there is a subsequence* $(W_{n_k})_k$ *of* $(W_n)_n$ *such that*

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_k)_k$ *of graphons the following conditions are equivalent:*

- $\mathrm{ACC}_{w*}((W_k)_k) = \mathrm{LIM}_{w*}((W_k)_k)$,
- $(W_k)_k$ *is Cauchy for the cut distance* $\delta_\square$.

Our proof of compactness

Key Theorem A

For every sequence $(W_n)_n$ of graphons there is a subsequence $(W_{n_k})_k$ of $(W_n)_n$ such that

$$\mathrm{ACC}_{w*}((W_{n_k})_k) = \mathrm{LIM}_{w*}((W_{n_k})_k).$$

Key Theorem B

For every sequence $(W_k)_k$ of graphons the following conditions are equivalent:

- $\mathrm{ACC}_{w*}((W_k)_k) = \mathrm{LIM}_{w*}((W_k)_k)$,
- $(W_k)_k$ *is Cauchy for the cut distance* $\delta_\square$.

If one of these conditions holds then $(W_k)_k$ converges in the cut distance $\delta_\square$ to the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subseteq \langle W \rangle$.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subseteq \langle W \rangle$.

Consider the mapping $W \mapsto \langle W \rangle$

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subseteq \langle W \rangle$.

Consider the mapping $W \mapsto \langle W \rangle$

- defined on the space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subseteq \langle W \rangle$.

Consider the mapping $W \mapsto \langle W \rangle$

- defined on the space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$
- with values in the hyperspace of all weak* compact subsets of $L^\infty([0,1]^2)$ equipped by the Vietoris topology.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subseteq \langle W \rangle$.

Consider the mapping $W \mapsto \langle W \rangle$

- defined on the space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$
- with values in the hyperspace of all weak* compact subsets of $L^\infty([0,1]^2)$ equipped by the Vietoris topology.

It turns out that it is a homeomorphism onto a closed subset of the hyperspace.

Structuredness order

What does it mean to be the 'most structured' element of $\mathrm{LIM}_{w*}((W_k)_k)$?

For every graphon W we define the envelope of W as $\langle W \rangle := \mathrm{LIM}_{w*}((W)_n)$.

We say that U is at most as structured as W, $U \preceq W$, if $\langle U \rangle \subseteq \langle W \rangle$.

Consider the mapping $W \mapsto \langle W \rangle$

- defined on the space of all (equivalence classes of) graphons equipped by the cut distance $\delta_\square$
- with values in the hyperspace of all weak* compact subsets of $L^\infty([0,1]^2)$ equipped by the Vietoris topology.

It turns out that it is a homeomorphism onto a closed subset of the hyperspace. As the hyperspace is compact, the space of all (equivalence classes of) graphons is compact as well.

How to find the most structured graphon?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

How to find the most structured graphon?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

Fix an arbitrary strictly convex function $f\colon [0,1] \to \mathbb{R}$.

How to find the most structured graphon?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

Fix an arbitrary strictly convex function $f\colon [0,1] \to \mathbb{R}$.

The most structured W in $\mathrm{LIM}_{w*}((W_n)_n)$ is that one which maximizes $\int_{[0,1]^2} f(W(x,y))$.

How to find the most structured graphon?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

Fix an arbitrary strictly convex function $f\colon [0,1] \to \mathbb{R}$.

The most structured W in $\mathrm{LIM}_{w*}((W_n)_n)$ is that one which maximizes $\int_{[0,1]^2} f(W(x,y))$.

In particular, if we choose choose $f(z) = z^2$ then we have that...

How to find the most structured graphon?

Suppose that $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

Fix an arbitrary strictly convex function $f\colon [0,1] \to \mathbb{R}$.

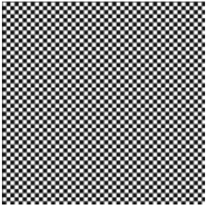
The most structured W in $\mathrm{LIM}_{w*}((W_n)_n)$ is that one which maximizes $\int_{[0,1]^2} f(W(x,y))$.

In particular, if we choose choose $f(z) = z^2$ then we have that...

...the most structured W in $\mathrm{LIM}_{w*}((W_n)_n)$ is that one which maximizes $\|W\|_{L^2}$.

Basic example once more

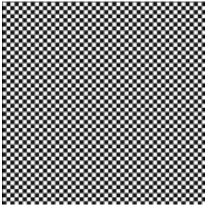
A representation of $K_{n,n}$ (for large n):



W_n

Basic example once more

A representation of $K_{n,n}$ (for large n):

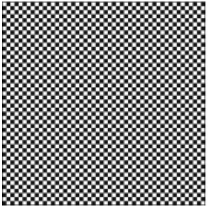


W_n

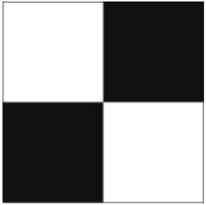
Then $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.

Basic example once more

A representation of $K_{n,n}$ (for large n):



W_n

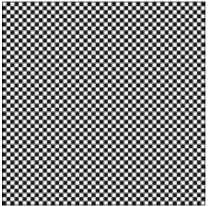


W_1

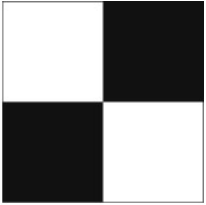
Then $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.
The constant graphon $C_{\frac{1}{2}} \equiv \frac{1}{2}$ and the graphon W_1 are both elements of $\mathrm{LIM}_{w*}((W_n)_n)$.

Basic example once more

A representation of $K_{n,n}$ (for large n):



W_n



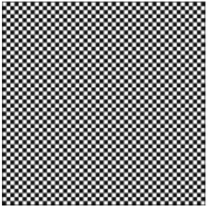
W_1

Then $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.
The constant graphon $C_{\frac{1}{2}} \equiv \frac{1}{2}$ and the graphon W_1 are both elements of $\mathrm{LIM}_{w*}((W_n)_n)$.
The graphon W_1 is more structured than $C_{\frac{1}{2}}$ as

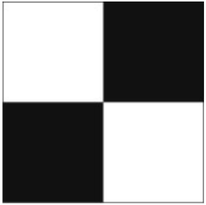
$$\int_{[0,1]^2} f(C_{\frac{1}{2}}(x,y)) = f\left(\tfrac{1}{2}\right) < \frac{1}{2}(f(0) + f(1)) = \int_{[0,1]^2} f(W_1(x,y)).$$

Basic example once more

A representation of $K_{n,n}$ (for large n):



W_n



W_1

Then $\mathrm{ACC}_{w*}((W_n)_n) = \mathrm{LIM}_{w*}((W_n)_n)$.
The constant graphon $C_{\frac{1}{2}} \equiv \frac{1}{2}$ and the graphon W_1 are both elements of $\mathrm{LIM}_{w*}((W_n)_n)$.
The graphon W_1 is more structured than $C_{\frac{1}{2}}$ as

$$\int_{[0,1]^2} f(C_{\frac{1}{2}}(x,y)) = f\left(\tfrac{1}{2}\right) < \frac{1}{2}(f(0) + f(1)) = \int_{[0,1]^2} f(W_1(x,y)).$$

In fact, W_1 is the most structured element of $\mathrm{LIM}_{w*}((W_n)_n)$.

Selected references

[1] M. Doležal, J. Hladký. Cut-norm and entropy minimization over weak* limits. Preprint.

[2] M. Doležal, J. Grebík, J. Hladký, I. Rocha, V. Rozhoň. Cut distance identifying graphon parameters over weak* limits. Manuscript.

[3] M. Doležal, J. Grebík, J. Hladký, I. Rocha, V. Rozhoň. Relating the cut distance and the weak* topology for graphons. Manuscript.

[4] L. Lovász. Large networks and graph limits (2012).

[5] L. Lovász and B. Szegedy. Limits of dense graph sequences (2006).