Mathematical methods in fluid mechanics

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Example of a PDE system in fluid mechanics

Euler system of compressible barotropic fluid

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0 \text{ in } (0, T) \times \Omega$$

Impermeability boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \ \mathbf{u}(0, \cdot) = \mathbf{u}_0$$

Mathematical problems - facts

Absence of global-in-time smooth solutions...

Smooth solutions typically develop shocks in a finite time; this is true for a "generic" class of data.

Weak solutions

$$\int [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] = 0$$

$$\int [\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi}] = 0 \text{ for smooth } \varphi, \ \boldsymbol{\varphi}$$

Admissibility - energy inequality

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{u} \right] \boxed{\leq} 0$$

$$P(\varrho) = \varrho \int_1^\infty \frac{p(z)}{z^2} \, \mathrm{d}z$$

Some (more shocking) facts about shocks

Recent mathematical (exact) results

- The problem admits global in time (weak) solutions for any (smooth) initial data (good news!)
- There are infinitely many weak solutions for any (smooth) initial data (bad news!)
- There infinitely many physically admissible weak solutions (satisfying the energy inequality) for a large class of (not necessarily smooth) data (even worse!)
- There are smooth (Lipshitz) initial data for which the problem admits infinitely many admissible weak solutions (devastating news!)

What is a good weak solution?

Desired properties

• A weak solution exists globally in time for “any” choice of the initial state
• A weak solution can be identified as a limit of suitable approximate problems, e.g. by adding artificial viscosity
• The set of weak solutions is closed; a limit of a family of weak solutions is another weak solution
• A weak solution can be identified as a limit of a numerical scheme
• A weak solution is the most general object that enjoys the weak–strong uniqueness property

Weak strong uniqueness

A weak solution coincides with a strong (classical) solution as long as the latter exists

Even more general solutions?

Measure–valued solutions

$$\int [\langle \nu_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \nabla_x \varphi] = \mathcal{R}_1$$

$$\int [\langle \nu_{t,x}; \varrho \mathbf{u} \rangle \cdot \partial_t \boldsymbol{\varphi} + \langle \nu_{t,x}; \varrho \mathbf{u} \otimes \mathbf{u} \rangle : \nabla_x \boldsymbol{\varphi} + \langle \nu_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \boldsymbol{\varphi}] = \mathcal{R}_2$$

$$\int \left\langle \nu_{\tau,x}; \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right\rangle + \mathcal{D} = \int \left\langle \nu_{0,x}; \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right\rangle$$

Compatibility

$$\mathcal{R}_1 + \mathcal{R}_2 \preceq \mathcal{D}$$

Why to go measure–valued

Main advantages

- They capture singularities - oscillations in hyperbolic systems.
- The notion is easy to extend to viscous fluids described via the Navier–Stokes equations.
- They are the solutions generates by (some) numerical schemes
- **Weak – strong uniqueness** A measure valued solution coincides with the strong solution emanating from the same initial data as long as the latter exists

Compressible Navier-Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Isentropic EOS, Newton's rheological law

$$p(\varrho) = a\varrho^{\gamma}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

No-slip boundary conditions

$$\mathbf{u}|_{\partial \Omega} = 0$$

Numerical method [T. Karper]

FV framework

$$\text{regular tetrahedral mesh}, \ Q_h = \{ v \mid v = \text{piece-wise constant} \}$$

FE framework - Crouzeix - Raviart

$$V_h = \left\{ v \;\middle|\; v = \text{piece-wise affine}, \ \tilde{v}_\Gamma \text{ continuous on face } \Gamma \right\}$$

$$\tilde{v}_\Gamma \equiv \frac{1}{|\Gamma|} \int_\Gamma v \ \mathrm{dS}_x$$

Upwind discretization of convective terms

$$\langle h\mathbf{u}; \nabla_x \varphi \rangle_E \approx \sum_\Gamma \int_\Gamma \mathrm{Up}[h, \mathbf{u}][[\varphi]] \ \mathrm{dS}_x$$

Dissipative upwind operator

Upwind operator

$$\mathrm{Up}[r_h, \mathbf{u}_h] = \underbrace{\{r_h\} \langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma}_{\text{convective part}} - \frac{1}{2} \underbrace{\max\{h^\alpha; |\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma|\} [[r_h]]}_{\text{dissipative part}}$$

$$= \underbrace{r_h^{\mathrm{out}} [\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma]^- + r_h^{\mathrm{in}} [\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma]^+}_{\text{standard upwind}} - \frac{h^\alpha}{2} [[r_h]] \chi \left(\frac{\langle \mathbf{u}_h \cdot \mathbf{n} \rangle_\Gamma}{h^\alpha} \right)$$

Auxilliary function

$$\chi(z) = \begin{cases} 0 \text{ for } z < -1, \\ z + 1 \text{ if } -1 \leq z \leq 0 \\ 1 - z \text{ if } 0 < z \leq 1 \\ 0 \text{ for } z > 1 \end{cases}$$

Numerical scheme

Discrete time derivative - implicit scheme

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}$$

Continuity method

$$\int_{\Omega_h} D_t \varrho_h^k \phi \mathrm{dx} - \sum_{\Gamma \in \Gamma_{\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \, [[\phi]] \, \mathrm{dS}_x = 0$$

Momentum method

$$\int_{\Omega_h} D_t \left(\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle \right) \cdot \boldsymbol{\phi} \mathrm{dx} - \sum_{\Gamma \in \Gamma_{\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle, \mathbf{u}_h^k] \cdot [[\langle \boldsymbol{\phi} \rangle]] \, \mathrm{dS}_x$$
$$- \int_{\Omega_h} p(\varrho_h^k) \mathrm{div}_h \boldsymbol{\phi} \mathrm{dx}$$
$$+ \mu \int_{\Omega_h} \nabla_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\phi} \mathrm{dx} + \left(\frac{\mu}{3} + \eta \right) \int_{\Omega_h} \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\phi} \mathrm{dx} = 0$$

Convergence results for Karper's scheme

Convergence to weak solutions

Karper [2013]: Convergence to a weak solution if $\boxed{\gamma > 3}$

Error estimates

Gallouet, Herbin, Maltese, Novotný [2015]

Convergence to smooth solutions + error estimates if $\boxed{\gamma > 3/2}$, Ω a polyhedral domain

Convergence for general adiabatic coefficient

EF, M. Lukáčová/Medviďová [2016]

Let $\Omega \subset R^3$ be a smooth bounded domain. Let

$$1 < \gamma < 2, \ \Delta t \approx h, \ 0 < \alpha < 2(\gamma - 1).$$

Suppose that the initial data are smooth and that the compressible Navier-Stokes system admits a smooth solution in $[0, T]$ in the class

$$\varrho, \ \nabla_x \varrho, \ \mathbf{u}, \nabla_x \mathbf{u} \in C([0, T] \times \overline{\Omega})$$

$$\partial_t \mathbf{u} \in L^2(0, T; C(\overline{\Omega}; R^3)), \ \varrho > 0, \ \mathbf{u}|_{\partial\Omega} = 0.$$

Then

$$\varrho_h \to \varrho \text{ (strongly) in } L^\gamma((0, T) \times K)$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2((0, T) \times K; R^3)$$

for any compact $K \subset \Omega$.

General strategy

Basic properties of numerical scheme

Show stability, consistency, discrete energy inequality

Measure valued solutions

Show convergence of the scheme to a $\boxed{\textbf{measure}-\textbf{valued solution}}$

Weak-strong uniqueness

Use the weak-strong uniqueness principle in the class of measure-valued solutions. Strong and measure valued solutions emanating from the same initial data coincide as long as the latter exists

Corollary

Convergence of numerical solutions

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution

Singular limit problem

Scaled Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = 0$$

Incompressible (low Mach) limit - EF, Ch.Klingenberg, S.Markfelder[2017]

Convergence to the limit system

$$\mathrm{div}_x \mathbf{v} = 0, \ \ \partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

for well/ill prepared initial data.

Complete Euler system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0$$

$$\partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] + \mathrm{div}_x \left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \mathbf{u} \right)$$

$$+ \mathrm{div}_x (p(\varrho, \vartheta) \mathbf{u}) = 0$$

Entropy inequality (admissibility)

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) \geq 0$$

Constitutive relations

$$p = \varrho \vartheta, \ \ e = c_v \vartheta, \ \ s = \log(\vartheta^{c_v}) - \log(\varrho)$$