A topological approach to periodic oscillations related to the Liebau phenomenon

Milan Tvrdý

jointly with

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Institute of Mathematics
Academy of Sciences of the Czech Republic

Ariel, August 2014
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1. VALVELESS PUMPING
   (Liebau phenomena)
In 1954 G. Liebau showed experimentally that a periodic compression made on an asymmetric part of a fluid-mechanical model could produce the circulation of the fluid without the necessity of a valve to ensure a preferential direction of the flow.
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**DEFINITION**

Let \( T > 0 \), \( g: \mathbb{R}^3 \to \mathbb{R} \) and let \( e: \mathbb{R} \to \mathbb{R} \) be nonconstant and \( T \)-periodic. Then the equation

\[
x'' = g(x, x', e(t))
\]

generates a **\( T \)-periodically forced pump** if it has a \( T \)-periodic solution \( x \) such that

\[
g(\bar{x}, 0, \bar{e}) \neq 0,
\]

i.e. the mean value \( \bar{x} \) of \( x \) is not an equilibrium of \( x'' = g(x, x', \bar{e}) \).
1 tank - 1 pipe model

G. Propst (2006)

\( \rho \), ... density of the liquid (constant)
\( p(t) \), ... \( T \) – periodic pressure
\( g \), ... acceleration of gravity
\( r_0 \), ... friction coefficient
\( \zeta \), ... junction coefficient

\( A_P/A_T \), ... cross sections of pipe/tank
\( V_0 \), ... constant total volume of liquid
\( w = -\ell' \), ... velocity in the pipe

\[ A_P \ell(t) + A_T h(t) \equiv V_0 \quad \Longrightarrow \quad h(t) \equiv \frac{1}{A_T} (V_0 - A_P \ell(t)) . \]

Momentum balance with Poiseuille’s law and Bernoulli’s equation
1 tank - 1 pipe model

G. Propst (2006)

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\( p(t) \) \ldots \( T \) – periodic pressure
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\( w = -\ell' \) \ldots velocity in the pipe

\[
A_P \ell(t) + A_T h(t) \equiv V_0 \implies h(t) \equiv \frac{1}{A_T} \left( V_0 - A_P \ell(t) \right).
\]

**Momentum balance with Poiseuille’s law and Bernoulli’s equation \( \implies \)**

\[
\ell \ell'' + a \ell \ell' + b (\ell')^2 + c \ell = e(t),
\]

where
\[
T > 0, \quad a = \frac{r_0}{\rho} \geq 0, \quad b = \left( 1 + \frac{\zeta}{2} \right) \geq 3/2,
\]

\[
e(t) = \frac{g V_0}{A_T} - \frac{p(t)}{\rho} \text{ is } T \text{ – periodic, } 0 < c = \frac{g A_p}{A_T} < 1.
\]
First observations

This leads to singular periodic problem:

(1) \[ u'' + au' = \frac{1}{u} \left( e(t) - b (u')^2 \right) - c, \quad u(0) = u(T), \quad u'(0) = u'(T), \]

\[ T > 0, \quad a = \frac{r_0}{\rho} \geq 0, \quad b = \left( 1 + \frac{\zeta}{2} \right) \geq 3/2, \quad 0 < c = \frac{gA_p}{A_T} < 1, \quad e(t) = \frac{gV_0}{A_T} - \frac{p(t)}{\rho}. \]
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\]

Multiplying the equation by \( u \) and integrating over \([0, T]\) gives

**THEOREM 1**

(1) has a positive solution only if \( \bar{e} \geq 0 \) (i.e. \( \bar{p} \leq \rho g \frac{V_0}{A_T} \)).
First observations

This leads to singular periodic problem:

$$(1) \quad u'' + au' = \frac{1}{u} \left( e(t) - b(u')^2 \right) - c, \quad u(0) = u(T), \quad u'(0) = u'(T),$$

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Multiplying the equation by $u$ and integrating over $[0, T]$ gives

**THEOREM 1**

$(1)$ has a positive solution only if \( \overline{e} \geq 0 \) (i.e. \( \overline{p} \leq \rho \frac{V_0}{A_T} \)).

**THEOREM 2**

If $(1)$ has a positive solution, then it generates a $T$-periodically forced pump.
Examples

(E) \( u'' + ku = \frac{b}{u^{\lambda}} + e(t) \), \( u(0) = u(T) \), \( u'(0) = u'(T) \) \((b > 0, \ \lambda > 0, \ k \geq 0, \ e \in L_1[0, T])\)

has a solution if:

- \( k = 0, \ \lambda \geq 1, \ \bar{e} < 0 \) \[\text{[Lazer & Solimini]},\]
- \( k \neq \left(n \frac{\pi}{T}\right)^2 \) for all \( n \in \mathbb{N}, \ \lambda \geq 1, \ e \in C \) \[\text{[del Pino, Manásevich & Montero]}\]
- \( 0 < k < \left(\frac{\pi}{T}\right)^2, \ \lambda \geq 1, \ e \in L_\infty \) \[\text{[Omari & Ye]},\]
- \( k = 0, \ \bar{e} < 0, \ e_* := \inf \text{ess} \inf_{t \in [0, T]} e(t) > -\left(\frac{1}{T^2 \lambda b}\right)^{\lambda+1} (\lambda+1) b, \)

\( 0 < k < \left(\frac{\pi}{T}\right)^2, \ e_* := \inf \text{ess} \inf_{t \in [0, T]} e(t) > -\left(\frac{\pi^2 - T^2 k}{T^2 \lambda b}\right)^{\lambda+1} (\lambda+1) b \)

[supplementary results by Torres, Hakl & Torres, Chu & Franco et al.],

\( k = \left(\frac{\pi}{T}\right)^2, \ \inf \text{ess} \inf_{t \in [0, T]} e(t) > 0 \) \[\text{[Rachůnková, Tvrdý & Vrkoč]},\]

[supplementary results by Bonheure & De Coster, Chu & Torres et al.]
2. EXISTENCE OF A PERIODIC SOLUTION
Existence of a periodic solution

(1) \[ u'' + au' = \frac{1}{u} (e(t) - b(u')^2) - c, \quad u(0) = u(T), \quad u'(0) = u'(T), \]

**THEOREM 3**

**ASSUME:**
- \( a \geq 0, \quad b > 1, \quad c > 0, \)
- \( e \) is continuous and \( T \)-periodic on \( \mathbb{R}, \ e_* > 0, \)
- \( \frac{(b + 1)c^2}{4e_*} < \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4}. \)

**THEN:** (1) has a positive solution.
Existence of a periodic solution

\[(1) \quad u'' + a u' = \frac{1}{u} \left( e(t) - b (u')^2 \right) - c, \quad u(0) = u(T), \ u'(0) = u'(T), \]

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ASSUME:

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THEN: \( (1) \) has a positive solution.

DEFINITION

A \( T \)-periodic function \( \sigma_1 \in C^2[0, T] \) is a **lower function** for

\[ u'' + a u' = f(t, u), \quad u(0) = u(T), \ u'(0) = u'(T), \]

if

\[ \sigma_1''(t) + a \sigma_1'(t) \geq f(t, \sigma_1(t)) \quad \text{for} \quad t \in [0, T], \]

while an **upper function** is defined analogously, but with reversed inequality.
\( (1) \quad u'' + au' = \frac{1}{u} (e(t) - b(u')^2) - c, \quad u(0) = u(T), \quad u'(0) = u'(T), \)

**STEP 1:** \( u : [0, T] \rightarrow \mathbb{R} \) is a positive solution of (1) iff \( x = u^{1/\mu} \) is a positive solution of

\( (2) \quad x'' + ax'(t) = r(t)x^\alpha - s(t)x^\beta, \quad x(0) = x(T), \quad x'(0) = x'(T), \)

where

\[ 0 < \mu = \frac{1}{b+1} < \frac{2}{5}, \quad r(t) = \frac{e(t)}{\mu} > 0, \quad s(t) = \frac{c}{\mu} > 0, \quad 0 < \alpha = 1 - 2\mu, \quad \beta = 1 - \mu < 1. \]
\( u'' + au' = \frac{1}{u}(e(t) - b(u')^2) - c, \quad u(0) = u(T), \ u'(0) = u'(T), \)

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\]

**STEP 2:** There are constant lower and upper functions \( \sigma_1 \) and \( \sigma_2 \) of (2) such that
\[
0 < \sigma_2 < x_0 = (r_*/s^*)^{1/(\beta-\alpha)} < x_1 = (r_*/s^*)^{1/(\beta-\alpha)} < \sigma_1.
\]
\( u'' + au' = \frac{1}{u} (e(t) - b(u')^2) - c \), \( u(0) = u(T), \ u'(0) = u'(T) \),

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\( x'' + ax'(t) = r(t) x^\alpha - s(t) x^\beta \), \( x(0) = x(T), \ x'(0) = x'(T) \),

where \( 0 < \mu = \frac{1}{b+1} < \frac{2}{5} \), \( r(t) = \frac{e(t)}{\mu} > 0 \), \( s(t) = \frac{c}{\mu} > 0 \), \( 0 < \alpha = 1 - 2\mu \), \( < \beta = 1 - \mu < 1 \).

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\( 0 < \sigma_2 < x_0 = (r*/s*)^{1/(\beta-\alpha)} < x_1 = (r*/s*)^{1/(\beta-\alpha)} < \sigma_1 \).

**STEP 3:** We show that there is \( \delta_0 \in (0, \sigma_2) \) such that

\( r(t) x^\alpha - s(t) x^\beta < 0 \) for \( t \in [0, T], \ x \in (0, \delta_0) \)

and

\[- \left( \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4} \right) x + r(t) x^\alpha - s(t) x^\beta < 0 \] for \( t \in [0, T], \ x \geq \delta_0 \).
\[ u'' + a u' = \frac{1}{u} (e(t) - b (u')^2) - c, \quad u(0) = u(T), \; u'(0) = u'(T), \]

**STEP 1:** \( u : [0, T] \rightarrow \mathbb{R} \) is a positive solution of (1) iff \( x = u^{1/\mu} \) is a positive solution of

\[ x'' + a x'(t) = r(t) x^\alpha - s(t) x^\beta, \quad x(0) = x(T), \; x'(0) = x'(T), \]

where

\[ 0 < \mu = \frac{1}{b+1} < \frac{2}{5}, \; r(t) = \frac{e(t)}{\mu} > 0, \; s(t) = \frac{c}{\mu} > 0, \; 0 < \alpha = 1 - 2 \mu, \; < \beta = 1 - \mu < 1. \]

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\[ 0 < \sigma_2 < x_0 = \left(\frac{r^*}{s^*}\right)^{1/(\beta - \alpha)} < x_1 = \left(\frac{r^*}{s^*}\right)^{1/(\beta - \alpha)} < \sigma_1. \]

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\[ r(t) x^\alpha - s(t) x^\beta < 0 \quad \text{for} \; t \in [0, T], \; x \in (0, \delta_0) \]

and

\[ -\left(\left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}\right) x + r(t) x^\alpha - s(t) x^\beta < 0 \quad \text{for} \; t \in [0, T], \; x \geq \delta_0. \]

**STEP 4:** We choose \( \delta \in (0, \delta_0) \), put \( \lambda^* = \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \),

\[ \tilde{f}(t, x) = \begin{cases} r(t) \delta^\alpha - s(t) \delta^\beta - \lambda^* (x - \delta) & \text{for} \; x < \delta, \\ r(t) x^\alpha - s(t) x^\beta & \text{for} \; x \geq \delta \end{cases} \]

and consider auxiliary problem

\[ (\text{Aux}) \quad x'' + a x'(t) = \tilde{f}(t, x), \quad x(0) = x(T), \; x'(0) = x'(T), \]
(1) \[ u'' + a u' = \frac{1}{u} \left( e(t) - b (u')^2 \right) - c, \quad u(0) = u(T), \quad u'(0) = u'(T), \]

**STEP 1:** \( u: [0, T] \rightarrow \mathbb{R} \) is a positive solution of (1) iff \( x = u^{1/\mu} \) is a positive solution of

(2) \[ x'' + a x' = r(t) x^\alpha - s(t) x^\beta, \quad x(0) = x(T), \quad x'(0) = x'(T), \]

where

\[ 0 < \mu = \frac{1}{b+1} < \frac{2}{5}, \quad r(t) = \frac{e(t)}{\mu} > 0, \quad s(t) = \frac{c}{\mu} > 0, \quad 0 < \alpha = 1 - 2 \mu, \quad < \beta = 1 - \mu < 1. \]

**STEP 2:** There are constant lower and upper functions \( \sigma_1 \) and \( \sigma_2 \) of (2) such that

\[ 0 < \sigma_2 < x_0 = (r_*/s_*)^{1/(\beta - \alpha)} < x_1 = (r_*/s_*)^{1/(\beta - \alpha)} < \sigma_1. \]

**STEP 3:** We show that there is \( \delta_0 \in (0, \sigma_2) \) such that

\[ r(t) x^\alpha - s(t) x^\beta < 0 \quad \text{for} \quad t \in [0, T], \quad x \in (0, \delta_0) \]

and

\[ \left( \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4} \right) x - r(t) x^\alpha - s(t) x^\beta < 0 \quad \text{for} \quad t \in [0, T], \quad x \geq \delta_0. \]

**STEP 4:** We choose \( \delta \in (0, \delta_0) \), put \( \lambda^* = \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4} \),

\[ \widetilde{f}(t, x) = \begin{cases} r(t) \delta^\alpha - s(t) \delta^\beta - \lambda^* (x - \delta) & \text{for} \ x < \delta, \\ r(t) x^\alpha - s(t) x^\beta & \text{for} \ x \geq \delta \end{cases} \]

and consider auxiliary problem

(\text{Aux}) \[ x'' + a x' = \widetilde{f}(t, x), \quad x(0) = x(T), \quad x'(0) = x'(T), \]

Method of non-ordered lower and upper functions (BONHEURE & De COSTER)

\( \implies \) (Aux) has a solution \( x \).
Sketch of the proof

**Steps 1–4:**

1. \( u'' + au' = \frac{1}{u} \left( e(t) - b(u')^2 \right) - c, \quad u(0) = u(T), \quad u'(0) = u'(T), \quad \updownarrow \)

2. \( x'' + ax'(t) = r(t)x^\alpha - s(t)x^\beta, \quad x(0) = x(T), \quad x'(0) = x'(T), \quad \)

where \( 0 < \mu = \frac{1}{b+1} < \frac{2}{5}, \quad r(t) = \frac{e(t)}{\mu} > 0, \quad s(t) = \frac{c}{\mu} > 0, \quad 0 < \alpha = 1 - 2\mu, \quad < \beta = 1 - \mu < 1. \)

We have a solution \( x \) to

(Aux) \( x'' + ax'(t) = \tilde{f}(t, x), \quad x(0) = x(T), \quad x'(0) = x'(T), \quad \)

where

\[
\tilde{f}(t, x) = \begin{cases} 
  r(t) \delta^\alpha - s(t) \delta^\beta - \lambda^* (x - \delta) & \text{for } x < \delta, \\
  r(t)x^\alpha - s(t)x^\beta & \text{for } x \geq \delta 
\end{cases}
\]

**Step 5:** Put \( v = x - \delta \). Then

\( v''(t) + av'(t) + \lambda^* v(t) = h(t) \quad \text{for } t \in [0, T], \quad v(0) = v(T), \quad v'(0) = v'(T), \quad \)

where (by Step 3) \( h(t) := \lambda^* (x(t) - \delta) - \tilde{f}(t, x(t)) \geq 0 \quad \text{on } [0, T]. \)
Sketch of the proof

Steps 1–4:

(1) \[ u'' + a u' = \frac{1}{u} \left( e(t) - b(u')^2 \right) - c, \quad u(0) = u(T), \quad u'(0) = u'(T), \]

$\Downarrow$

(2) \[ x'' + a x'(t) = r(t) x^\alpha - s(t) x^\beta 0, \quad x(0) = x(T), \quad x'(0) = x'(T), \]

where

\[ 0 < \mu = \frac{1}{b+1} < \frac{2}{5}, \quad r(t) = \frac{e(t)}{\mu} > 0, \quad s(t) = \frac{c}{\mu} > 0, \quad 0 < \alpha = 1 - 2 \mu, \quad \beta = 1 - \mu < 1. \]

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where (by Step 3) \( h(t) := \lambda^* (x(t) - \delta) - \tilde{f}(t, x(t)) \geq 0 \) on \([0, T]\).

Antimaximum principle \( \text{ (OMARI & TROMBETTA or HAKL & ZAMORA) } \implies \quad v \geq 0, \quad \text{i.e. } x \geq \delta \quad \square \)
Existence of a periodic solution

(2) \[ u'' + au' = r(t) u^\alpha - s(t) u^\beta, \quad u(0) = u(T), \quad u'(0) = u'(T), \]

**THEOREM 4**

**Assume:**
- \( a \geq 0, \quad b > 1, \quad c > 0, \quad 0 < \alpha < \beta < 1, \)
- \( r_* > 0, \quad s_* > 0, \)
- there is \( \delta_0 > 0 \) such that
  \[ r(t) u^\alpha - s(t) u^\beta < 0 \quad \text{for} \quad t \in [0, T], \quad x \in (0, \delta_0) \]
  and
  \[ - \left( \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4} \right) x + r(t) x^\alpha - s(t) x^\beta < 0 \quad \text{for} \quad t \in [0, T], \quad x \geq \delta_0. \]

**Then:** (2) has a positive solution.
3. ASYMPTOTIC STABILITY
Asymptotic stability

(3) \[ x'' + a x'(t) = f(t, x), \quad x(0) = x(T), \, x'(0) = x'(T) \]

**Lemma** (Omari & Njoku, 2003)

**Assume:** \( a > 0, \)

- \( \sigma_1 \) is a strict lower function, \( \sigma_2 \) is a strict upper function of (3) and \( \sigma_2 < \sigma_1 \) on \([0, T] \).
- \[ \frac{\partial}{\partial x} f(t, x) \geq - \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4} \quad \text{for} \quad t \in [0, T], \, x \in [\sigma_2(t), \sigma_1(t)], \]
- there is a continuous \( \gamma : [0, T] \to [0, \infty) \) such that \( \tilde{\gamma} > 0 \) and \( \frac{\partial}{\partial x} f(t, x) \leq - \gamma(t) \quad \text{for} \quad t \in [0, T], \, x \in [\sigma_2(t), \sigma_1(t)]. \)

Then (3) has at least one asymptotically stable \( T \)-periodic solution \( x \) fulfilling \( \sigma_2 \leq x \leq \sigma_1 \) on \([0, T] \).
\[ x'' + ax'(t) = f(t, x), \quad x(0) = x(T), \ x'(0) = x'(T) \]

**THEOREM 5**

**Assume:** \( a > 0, \ f(t, x) = r(t) x^\alpha - s(t) x^\beta, \)

- \( r, s \) are continuous and positive on \([0, T], 0 < \alpha < \beta < 1, \)
- \( \beta s^* \left( \frac{s^*}{r^*} \right)^{(1-\beta)/(\beta-\alpha)} - \alpha r^* \left( \frac{s^*}{r^*} \right)^{(1-\alpha)/(\beta-\alpha)} < \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4}, \)
- \( \frac{\alpha}{\beta} \frac{r^*}{s^*} < \frac{r^*}{s^*}. \)

**Then:** (3) has at least one asymptotically stable positive solution.
THEOREM 5

Assume: $a > 0$, $f(t, x) = r(t) x^\alpha - s(t) x^\beta$,

- $r, s$ are continuous and positive on $[0, T]$, $0 < \alpha < \beta < 1$,
- $\beta s^* \left( \frac{s^*}{r^*} \right)^{(1-\beta)/(\beta-\alpha)} - \alpha r^* \left( \frac{s^*}{r^*} \right)^{(1-\alpha)/(\beta-\alpha)} < \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4}$,
- $\frac{\alpha}{\beta} \frac{r^*}{s^*} < \frac{r^*}{s^*}$.

Then: (3) has at least one asymptotically stable positive solution.

COROLLARY

(1) has at least one asymptotically stable positive solution if

$$\frac{c^2 \left( b (e^*)^2 - (b - 1) (e^*)^2 \right)}{e^* (e^*)^2} < \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4} \quad \text{and} \quad \frac{e^* - e^*}{e^*} < \frac{1}{b}.$$
4. APPLICATION OF KRASNOSELSKIđII
COMPRESION/EXPANSION THEOREM
\[ x'' + ax' + m^2 x = 0, \quad x(0) - x(T), \quad x'(0) = x'(T) \quad \begin{bmatrix} a \geq 0, & 0 < m^2 < \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \end{bmatrix} \]
(4) \( x'' + a x' + m^2 x = 0, \ x(0) - x(T), \ x'(0) = x'(T) \)

\[ a \geq 0, \ 0 < m^2 < \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \]

has Green's function \( G_m(t, s) \) such that

- \( G_m(t, s) > 0 \) for all \( t, s \in [0, T] \),
- there exists \( c_m \in (0, 1) \) such that \( G_m(s, s) \geq c_m G_m(t, s) \) for all \( t, s \in [0, T] \).
(4) \( x'' + ax' + m^2 x = 0, \ x(0) - x(T), \ x'(0) = x'(T) \) \( a \geq 0, \ 0 < m^2 < \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \) has Green's function \( G_m(t, s) \) such that

- \( G_m(t, s) > 0 \) for all \( t, s \in [0, T] \),
- there exists \( c_m \in (0, 1) \) such that \( G_m(s, s) \geq c_m G_m(t, s) \) for all \( t, s \in [0, T] \),

Put \( (Fx)(t) = \int_0^T G_m(t, s) \left[ r(s)x^\alpha(s) - s(t)x^\beta(s) + m^2 x(s) \right] ds \)
(4) \( x'' + ax' + m^2 x = 0, \ x(0) - x(T), \ x'(0) = x'(T) \quad \left[ a \geq 0, \ 0 < m^2 < \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \right] \)

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Put \( (Fx)(t) = \int_0^T G_m(t, s) \left[ r(s) x^\alpha(s) - s(t) x^\beta(s) + m^2 x(s) \right] ds \)

Then \( x \) is a solution to

(2) \( x'' + ax' = r(t) x^\alpha - s(t) x^\beta, \quad x(0) = x(T), \ x'(0) = x'(T) \)

iff \( x = Fx \).
(4) \[ x'' + ax' + m^2x = 0, \quad x(0) - x(T), \quad x'(0) = x'(T) \quad \left[ a \geq 0, \quad 0 < m^2 < \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \right] \]

has Green's function \( G_m(t, s) \) such that
- \( G_m(t, s) > 0 \) for all \( t, s \in [0, T] \),
- there exists \( c_m \in (0, 1) \) such that \( G_m(s, s) \geq c_m G_m(t, s) \) for all \( t, s \in [0, T] \),

Put \( (Fx)(t) = \int_0^T G_m(t, s) \left[ r(s) x^\alpha(s) - s(t) x^\beta(s) + m^2 x(s) \right] ds \)

Then \( x \) is a solution to
\[
(2) \quad x'' + ax' = r(t) x^\alpha - s(t) x^\beta, \quad x(0) = x(T), \quad x'(0) = x'(T)
\]

iff \( x = Fx \).

**Krasnoselskii Fixed Point Theorem**

Let \( P \) be a cone in \( X \), \( \Omega_1 \) and \( \Omega_2 \) be bounded open sets in \( X \) such that \( 0 \in \Omega_1 \) and \( \overline{\Omega}_1 \subset \Omega_2 \). Let \( F : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P \) be a completely continuous operator such that one of the following conditions holds:
- \( \|Fx\| \geq \|x\| \) for \( x \in P \cap \partial \Omega_1 \) and \( \|Fx\| \leq \|x\| \) for \( x \in P \cap \partial \Omega_2 \),
- \( \|Fx\| \leq \|x\| \) for \( x \in P \cap \partial \Omega_1 \) and \( \|Fx\| \geq \|x\| \) for \( x \in P \cap \partial \Omega_2 \).

Then \( F \) has a fixed point in the set \( P \cap (\overline{\Omega}_2 \setminus \Omega_1) \).
\[ (2) \quad x'' + ax' = r(t)x^\alpha - s(t)x^\beta, \quad x(0) = x(T), \ x'(0) = x'(T) \]

- \( G_m(t, s) > 0 \) for all \( t, s \in [0, T] \),
- there exists \( c_m \in (0, 1) \) such that \( G_m(s, s) \geq c_m G_m(t, s) \) for all \( t, s \in [0, T] \),

Put
- \( P = \{ x \in C[0, T]: x(t) \geq 0 \text{ on } [0, T] \text{ and } x(t) \geq c_m \|x\| \text{ on } [0, T] \} \),
- \( \Omega_1 = \{ x \in C[0, T]: \|x\| < R_1 \} \), \( \Omega_2 = \{ x \in C[0, T]: \|x\| < R_2 \} \).
(2) \[ x'' + ax' = r(t)x^\alpha - s(t)x^\beta, \quad x(0) = x(T), \quad x'(0) = x'(T) \]

- \( G_m(t, s) > 0 \) for all \( t, s \in [0, T] \),
- there exists \( c_m \in (0, 1) \) such that \( G_m(s, s) \geq c_m G_m(t, s) \) for all \( t, s \in [0, T] \),

Put

- \( P = \{ x \in C[0, T] : x(t) \geq 0 \text{ on } [0, T] \text{ and } x(t) \geq c_m \|x\| \text{ on } [0, T] \} \),
- \( \Omega_1 = \{ x \in C[0, T] : \|x\| < R_1 \} \), \( \Omega_2 = \{ x \in C[0, T] : \|x\| < R_2 \} \).

**THEOREM 6**

**ASSUME:** \( a \geq 0, \ r, s \in C[0, T], \ 0 < \alpha < \beta < 1, \)

there exist \( m > 0 \) and \( 0 < R_1 < R_2 \) such that \( m^2 < \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \),

\[
\begin{align*}
    r(t)x^\alpha - s(t)x^\beta + m^2x &\geq 0 & \text{for } t \in [0, T], \ x \in [c_m R_1, R_2], \\
    r(t)x^\alpha - s(t)x^\beta + m^2x &\geq m^2 R_1 & \text{for } t \in [0, T], \ x \in [c_m R_1, R_1], \\
    r(t)x^\alpha - s(t)x^\beta + m^2x &\leq m^2 R_2 & \text{for } t \in [0, T], \ x \in [c_m R_2, R_2],
\end{align*}
\]

**THEN:** (2) has a positive solution \( x \in [c_m R_1, R_2] \).
Application of Krasnoselskii compression/expansion theorem

(2) \[ x'' + ax' = r(t) x^\alpha - s(t) x^\beta, \quad x(0) = x(T), \quad x'(0) = x'(T) \]

**COROLLARY = THEOREM 3**

**Assume:**
- \( a \geq 0, \quad b > 1, \quad c > 0, \)
- \( e \) is continuous and \( T \)-periodic on \( \mathbb{R}, \quad e_* > 0, \)
- \[ \frac{(b + 1)c^2}{4e_*} < \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4}. \]

**Then:** (1) has a positive solution.

**Remark**

Compare conditions:
- **Theorem 3:** there is \( \delta > 0 \) such that
  \[ \left( \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \right) x - f(t, x) \geq \left( \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \right) \delta \quad \text{for} \ t \in [0, T], \ x \geq \delta, \]

- **Theorem 6:** there is \( m \in \left( 0, \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \right), \) such that
  \[ m^2 x - f(t, x) \geq 0 \quad \text{for} \ t \in [0, T], \ x \in [c_m R_1, R_2] \]

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