Decentralized Control of Product (max+)-automata using Coinduction

Jan Komenda, Sébastien Lahaye, and Jean-Louis Boimond

Institute of Mathematics, Czech Academy of Sciences,
Brno, Czech Republic
and
LISA, ISTIA
Angers, France

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2. Preliminaries from co/algebra and supervisory control
3. Centralized control using coalgebra
4. Decentralized control
5. CONCLUDING REMARKS
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Control of \((\text{max,}+)\) automata inspired by supervisory control

- \((\text{Max,}+)\) automata: weighted automata with weights in 
  \(\overline{\mathbb{R}}_{\text{max}} = (\mathbb{R} \cup \{-\infty\}, \max, +)\).
- class of **Timed Discrete Event (dynamical) Systems (TDES)** with synchronization and resource sharing
- synchronous composition of \((\text{max,}+)\)-automata: extended alphabet or non determinism
- Definitions by Coinduction of synchronous and supervised product
- Proofs by Coinduction of theorems modular synthesis equals global synthesis
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Decentralized Control of Product (max+)-automata

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(max,+)-automata generalize both logical automata and (max,+)-linear systems (e.g. timed event graphs).

(max,+)-automata are quadruples $G = (Q, A, q_0, t)$, where
- $Q$ is the set of states,
- $q_0$ is the initial state,
- $A$ is the set of discrete events,
- $t : Q \times A \times Q \rightarrow \mathbb{R}_{max}$ is the transition function.

**Meaning:** output value $t(q, a, q') \in \mathbb{R}_{max}$ corresponds to the $a$-transition from $q$ to $q'$ and $t(q, a, q') = \varepsilon$ if there is no transition from $q$ to $q'$ labeled by $a$.

Deterministic (max,+)-automata: $t$ is deterministic, i.e. $t : Q \times A \rightarrow Q \times \mathbb{R}_{max}$.
Deterministic (max,+)-automata as coalgebras

- Deterministic (max,+)-automata are coalgebras 
  \((S, t)\), where \(S\) is the set of states and the transition function is 
  \(t : S \to (1 + (\mathbb{R}_{\max} \times S))^A\) with \(1 = \{\emptyset\}\).
- Their behaviors are stream functions \(f : A^\omega \to \mathbb{R}_{\max}^\omega\). 
  \(f : A^\omega \to \mathbb{R}_{\max}^\omega\) is causal if \(\forall n \in \mathbb{N}, \sigma, \tau \in A^\infty : \forall i : i \leq n:\) 
  \(\sigma(i) = \tau(i)\) then \(f(\sigma)(n) = f(\tau)(n)\).
- Stream derivatives: \(\omega = (\omega_0, \omega_1, \ldots) \in K^\omega \to \omega' = (\omega_1, \omega_2, \ldots)\).
- Stream functions form final coalgebra of (max,+)-automata with 
  \(t(f) = \langle f[a], f_a \rangle \) if \(f[a] \neq \emptyset \in 1\), 
  \(\emptyset\) otherwise, 
  \(f[a] = f(a : \sigma)(0)\) and \(f_a(\sigma) = f(a : \sigma)'\)
- \(A^\infty = A^\omega \cup A^+\), where \(A^+ = A^* \setminus \{\lambda\}\)
- \(f\) is consistent if \(\sigma \in A^\omega : f(\sigma)(k) = \emptyset \Rightarrow f(\sigma)(n) = \emptyset \forall n > k\).

**Theorem. (Rutten 2006)**

\(\mathcal{F} = (\mathcal{F}, t_{\mathcal{F}})\) is the final coalgebra of (max,+)-automata:

\(\mathcal{F} = \{ f : A^\omega \to (1 + K)^\omega \mid f\ \text{is causal and consistent}\}\).

\(t_{\mathcal{F}}(f)(a) = \begin{cases} 
\langle f[a], f_a \rangle & \text{if } f[a] \neq \emptyset \in 1, \\
\emptyset & \text{otherwise},
\end{cases}\)
Equivalent presentation of behaviors

- $S_0 \xrightarrow{\sigma(0)} S_1 \xrightarrow{\sigma(1)} S_2 \cdots \xrightarrow{\sigma(n)} S_{n+1}$.
  We define $l(S_0)(\sigma)(n) = k_n$.

- $\mathcal{F}$ is isomorphic to functions between finite and infinite sequences!

  $\mathcal{F}_\infty = \{ f : A^\infty \rightarrow \overline{\mathbb{R}}_{\text{max}}^\infty \mid f \text{ preserve length, causal, } \& \text{dom}(f)\text{prefix-closed} \}$

- $f[a] = f(a)(0)$ whenever $f$ is defined for $a \in A$.

- $f_a : A^\infty \rightarrow (1 + \overline{\mathbb{R}}_{\text{max}})^\infty$ given by $f_a(s) = f(a : s)$

  $t_{\mathcal{F}_\infty}(f)(a) = \begin{cases} \langle f[a], f_a \rangle & \text{if } f[a] \text{ is defined} \\ \text{undefined} & \text{otherwise,} \end{cases}$
Residuation theory

**Residuation theory** generalizes inversion
An isotone \( f : \mathcal{D} \rightarrow \mathcal{C} \), where \( \mathcal{D} \) and \( \mathcal{C} \) are dioids (naturally ordered \( a \preceq b \) iff \( a \oplus b = b \)), is said to be **residuated** if there exists an isotone map \( h : \mathcal{C} \rightarrow \mathcal{D} \) such that

\[
f \circ h \preceq \text{Id}_\mathcal{C} \quad \text{and} \quad h \circ f \succeq \text{Id}_\mathcal{D}.
\]

\( h \) is unique residual of \( f \), denoted by \( f^\# \).

If \( f \) is residuated then \( \forall y \in \mathcal{C}, \sup\{x \in \mathcal{D} | f(x) \preceq y\} \) exists and belongs to this subset and is equal to \( f^\#(y) \).

**Example:** left and right multiplications are always residuated in complete dioids!

**Notation.**

\[
a \searrow y = \max\{x | a \odot x \leq y\} \quad \text{and} \quad \quad y \nearrow a = \max\{x | x \odot a \leq y\}.
\]
Supervisory control

Control framework: Given two deterministic (max,+)-automata

\[ G_c = (Q_c, q_c, 0, Q_m, t_c), \quad G = (Q_g, q_g, 0, Q_m, t_g). \]

we consider their behaviors \( y_c \in \mathcal{F} \) and \( y \in \mathcal{F} \). Closed-loop system will be defined via \textit{supervised product}, denoted \( y^c \otimes_{A_u} y \)

Distinguish \( A_c \subseteq A \) is the subset of \textit{controllable events}, \( A_u = A \setminus A_c \) is the subset of \textit{uncontrollable events}.

Spec. \( y^{ref} \in \mathcal{F} \) is \textit{admissible} wrt \( y \in \mathcal{F} \) if \( L(y^{ref}) \subseteq L(y) \) and for all \( w \in L(y^{ref}) \) there is \( y^{ref}(w) \geq y(w) \) (meant component-wise).

Controller \( y^c \in \mathcal{F} \) is \textit{admissible} wrt \( y \in \mathcal{F} \) if \( L(y^c) \subseteq L(y) \) and \( \forall w \in L(y^{ref}) \) there is \( y^c(w) \geq 0 \) (meant component-wise).
Supervisory control: coalgebraic framework

**Notation.** \( y^{\text{ref}} : A^\infty \to (R_{\text{max}})^\infty \) is (an admissible) control specification

Natural order: for sequential functions \( y, y' : A^\infty \to K^\infty \) we write \( y \preceq y' \) iff \( L(y) \subseteq L(y') \) and \( \forall w \in L(y) : y(w) \leq y(w') \)

**Problem.** Find a greatest admissible controller \( y^c \) such that \( y^c \otimes_{A_u} y \preceq y^{\text{ref}} \).

**Admissible controller:** it does not disable nor delay uncontrollable events.

\( L(y^{\text{ref}}) \) is **controllable** wrt \( L(y) \) and \( A_u \) if

\[
\overline{L(y^{\text{ref}})} A_u \cap L(y) \subseteq \overline{L(y^{\text{ref}})}.
\]
Introduction

Preliminaries from co/algebra and supervisory control

Centralized control using coalgebra

Decentralized control

CONCLUDING REMARKS

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Supervised product by coinduction

**Definition (Supervised product).** Given System and Controller, resp. \( y, y^c \in \mathcal{F}, \forall a \in A \):

\[
(y^c \otimes_{A_u} y)_a = \begin{cases} 
  y^c_a \otimes_{A_u} y_a & \text{if } y^c \xrightarrow{\bar{a}} \text{ and } y \xrightarrow{\bar{a}} \\
  0 \otimes_{A_u} y_a & \text{if } a \in A_{uc} \text{ and } y^c \not\xrightarrow{\bar{a}} \text{ and } y \xrightarrow{\bar{a}} \\
  \emptyset & \text{otherwise}
\end{cases}
\]

and

\[
(y^c \otimes_{A_u} y)[a] = \begin{cases} 
  y^c[a] \otimes y[a] & \text{if } a \in A_c \\
  y[a] & \text{otherwise}
\end{cases}
\]
Example 1.

Then \( y(a(ub)^\omega) = (1(2, 1)^\omega) \).
Example 1.

Controller $y^c$ delays the first uncontrollable $u$, delays the first $b$, and tries to forbid the second $u$.

\[
(y^c \otimes_{A_u} y)[a] = y^c[a] \otimes y[a] = 2 \otimes 1 = 3,
\]
\[
(y^c \otimes_{A_u} y)[u] = y_a[u] = 2,
\]
\[
(y^c \otimes_{A_u} y)[u] = y^c_a[b] \otimes y_{au}[b] = 2 \otimes 1 = 3,
\]
\[
(y^c \otimes_{A_u} y)[u] = y_{au}[u] = 2.
\]

Figure: Closed-loop system automaton

Note that $(y^c \otimes_{A_u} y)_{aub} \xrightarrow{u} u$, because $u \in A_u$ and $y_{au} \xrightarrow{u} (even though y^c_{aub} \not\xrightarrow{u})$. 
Main result: least restrictive controller

Theorem 1. For any \( y, y^{ref} \in \mathcal{F} \) with \( y^{ref} \) admissible with respect to \( y \) we have:

\[
(y^{ref}/_{A_u} y)_a = \begin{cases} 
(y^{ref})_a/_{A_u} y_a & \text{if } C \\
\emptyset & \text{otherwise}
\end{cases}
\]

and

\[
(y^{ref}/_{A_u} y)[a] = \begin{cases} 
y^{ref}[a] \not\in y[a] & \text{if } a \in A_c \text{ and } C \\
y[a] & \text{if not } C \\
T & \text{if } a \in A_u \text{ and } C
\end{cases}
\]

where the auxiliary condition \( C \) is defined as

\( y^{ref} \xrightarrow{a} \) and \( y \xrightarrow{a} \) and \( \forall u \in A_u^* : y_a \xrightarrow{u} \Rightarrow y^{ref}_a \xrightarrow{u} \).
Example 1 continued.

Let $y(a(ub)^\omega) = (1(2, 1)^\omega)$.

![Specification automaton](image1)

**Figure:** Specification automaton

![Controller](image2)

**Figure:** Controller $(y_{\text{ref}}^{\text{ref}} / A_u y)$

![Closed-loop system](image3)

**Figure:** Closed-loop system $(y_{\text{ref}}^{\text{ref}} / A_u y) \otimes A_u y$
Extended alphabet  \( \mathcal{A} = (A_1 \cap A_2) \cup (A_1 \setminus A_2)^* \times (A_2 \setminus A_1)^* \)

For \( l_i \in \mathcal{F} \) over \( A_i \) and \( v_i = a_1 \ldots a_k \in A_i^{+} \) we define for \( i = 1, 2 \):

\[
    l_i[v_i] = (l_i)[a_1] \otimes (l_i)_{a_1}[a_2] \otimes \cdots \otimes (l_i)_{a_1 \ldots a_{k-1}}[a_k].
\]

Definition. for \( l_1, l_2 \in \mathcal{F} \) and \( \forall v \in \mathcal{A} \):

\[
    (l_1 \parallel l_2)_v = (l_1)_{P_1(v)} \parallel (l_2)_{P_2(v)} \quad \text{and} \quad (l_1 \parallel l_2)[v] = l_1[P_1(v)] \otimes B[l_2[P_2(v)]] \oplus B[l_1[P_1(v)]] \otimes l_2[P_2(v)].
\]
Synchronous product continued

\[(l_1 \| l_2)[v] = \begin{cases} 
\max(l_1[P_1(v)], l_2[P_2(v)]) & \text{if } l_i[P_i(v)] \neq \varepsilon \text{ for } i = 1, 2 \\
\varepsilon & \text{else, i.e. } \exists i = 1, 2 : l_i[P_i(v)] = \varepsilon
\end{cases}\]

Hint for understanding:
for partial languages \(L_1 = (L_1^1, L_1^2)\), \(L_2 = (L_2^1, L_2^2)\), and \(w \in A^*\)
we have in fact
\[(L_1 \| L_2)_w = (L_1)_{P_1(w)} \| (L_2)_{P_2(w)}.
\]
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Definition. \((P_1(\nu), P_2(\nu))\) is controllable iff there exists a controllable event in either \(P_1(\nu)\) or in \(P_2(\nu)\).

Equiv.: \((P_1(\nu), P_2(\nu)) \in \mathcal{A}_u\) is uncontrollable iff both local strings \(P_1(\nu) \in \mathcal{A}_{u,1}^*\) and \(P_2(\nu) \in \mathcal{A}_{u,2}^*\).

Problem. When does global control synthesis equal decentralized one?

The global control synthesis amounts to computing

\[(y_1^{\text{ref}} \parallel y_2^{\text{ref}}) / \# \mathcal{A}_u (y_1 \parallel y_2) \otimes_{\mathcal{A}_u} (y_1 \parallel y_2),\]

while the local control synthesis amounts to computing

\[[y_1^{\text{ref}} / \mathcal{A}_u y_1 \otimes_{\mathcal{A}_u} y_1] \parallel [y_2^{\text{ref}} / \mathcal{A}_u y_2 \otimes_{\mathcal{A}_u} y_2].\]
Main result: decentralized vs. global control

We say that:

- local subsystems agree on the controllability status of shared events if $A_{u,1} \cap A_2 = A_{u,2} \cap A_1$.
- Local languages $L_i$, $i = 1, 2$ are mutually controllable if $L_1$ is controllable with respect to $P_1P_2^{-1}(L_2)$ and $A_{u,1} \cap A_2$ and $L_2$ is controllable with respect to $P_2P_1^{-1}(L_1)$ and $A_1 \cap A_{u,2}$.
- local specifications do not require to delay locally uncontrollable events if for all $u_i \in A_{u,i}$ we have $y_i[u_i] = y_i^{ref}[u_i]$.

**Theorem.** Let $y = y_1 \parallel y_2 : A^\infty \rightarrow (\mathbb{R}_{\max})^\infty$ be the global behavior and $y^{ref} = y_1^{ref} \parallel y_2^{ref}$ the global specification. If the local languages $L(y_1)$ and $L(y_2)$ are mutually controllable, if the local subsystems agree on the controllability status of shared events and if local specifications do not require to delay locally uncontrollable events then

$$(y^{ref}/_{A_u} y) \otimes_{A_u} y = ([y_1^{ref}/_{A_{u,1}} y_1] \otimes_{A_u} y_1) \parallel ([y_2^{ref}/_{A_{u,2}} y_2] \otimes_{A_u} y_2).$$
Remark.

- Problematic case: a concurrent event composed of a controllable component event in the first component and an uncontrollable component event in the second component \((b, c) \in \mathcal{A})\).

- From a timing viewpoint decentralized control yields the first output equal to \(y_1^{\text{ref}}[b] \oplus y_2[c]\), while global control yields the first output equal to \(y_1^{\text{ref}}[b] \oplus y_2^{\text{ref}}[c]\).

- Note that due to admissibility of the (local) specifications we always have \(y_2[c] \leq y_2^{\text{ref}}[c]\) (the same as in the purely logical setting.).

- Conclusion: the same inequality holds in general as in the purely logical setting!
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Conclusion

- synchronous and supervised composition of deterministic (max,+)-automata by coinduction
- Supervisory control: residuation theory
- Centralized supervision: coinductive formula for
- Application to decentralized supervisory control of (max,+)-automata: sufficient conditions for local control synthesis equals global control synthesis
- Controllability and Supremal controllable (max,+) series
- More work on control of (max,+)-automata is needed: control with partial observations, coordination control.