AN ANALYTIC MODEL FOR COMMUTING OPERATOR TUPLES

Miroslav Engliš

(a joint work with C. Ambrozie and V. Müller)

Let T be a bounded linear operator on a separable Hilbert space H. A well-known result of Sz.-Nagy and Foias says that the following two assertions are equivalent:

(a) $I - TT^* \geq 0$ (i.e. T is a contraction) and $T^{*n} \to 0$ in the strong operator topology;

(b) T^* is unitarily equivalent to the restriction of a backward shift of infinite multiplicity to an invariant subspace.

In this talk we describe a generalization of this result for commuting n-tuples $T = (T_1, \dots, T_n)$ of operators on H.

Let Ω be a domain in $\mathbb{C}^n$ and $\mathcal{H}$ a Hilbert space of analytic functions on Ω which satisfies the following properties:

(1) $\mathcal{H}$ is invariant under the operators Z_j of multiplications by the coordinate functions ($j = 1, \dots, n$).

(2) The evaluation functionals are continuous on $\mathcal{H}$. Consequently, there exists a reproducing kernel $K(z, w)$ for $\mathcal{H}$.

(3) $\mathcal{H}$ contains all polynomials and they are dense in it.

(4) The function $1/K(z, w)$ is a polynomial (in z and $\overline{w}$).

It is well known that for any orthonormal basis $\{\psi_k\}$ of $\mathcal{H}$, the reproducing kernel is given by

$$K(z, w) = \sum_k \psi_k(z)\overline{\psi_k(w)}.$$

In view of (3), by applying the Gramm-Schmidt orthogonalization process, we may construct a basis such that all ψ_k are polynomials (and, conversely, any polynomial is a linear combination of a finite number of the ψ_k). We fix such a basis from now on. For each m, set

$$f_m(z, \overline{w}) = \sum_{k \geq m} \frac{\psi_k(z)\overline{\psi_k(w)}}{K(z, w)}.$$

Then $f_0(z, \overline{w}) \equiv 1$ on $\Omega \times \Omega$. By virtue of (4) and our choice of the basis, the difference $f_0 - f_m$ is a polynomial in $z, \overline{w}$, for each m; thus f_m themselves are, in fact, polynomials.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For any polynomial $p(z,\overline{w})$ of $z,\overline{w}\in\mathbb{C}^n$, define

$$p(T,T^*)=\sum_{\alpha,\beta}p_{\alpha\beta}T^\alpha T^{*\beta} \qquad \text{if} \qquad p(z,\overline{w})=\sum_{\alpha,\beta}z^\alpha\overline{w}^\beta.$$

(Up to the order of T and T^*, this coincides with the "hereditary calculus" of Agler [A1].) Hence, in particular, $f_m(T,T^*)$ are defined for any commuting operator tuple T, and $f_0(T,T^*)=I$. Our main result is the following.

Theorem. *Let $\mathcal{H},\psi_k,f_m$ be as above and let T be a commuting n-tuple of operators on a separable Hilbert space H. Denote by M_z the operator n-tuple $Z\otimes I$ on the Hilbert space tensor product $\mathcal{H}\otimes H$. Then the following are equivalent:*

(a) $\frac{1}{K}(T,T^*)\ge 0$ *and* $\langle f_m(T,T^*)h,h\rangle\to 0\ \forall h\in H$.
(b) T^* *is unitarily equivalent to a restriction of* M_z^* *to an invariant subspace.*

Example. Let $n=1$, $\Omega=\mathbb{D}$, the unit disc, and $\mathcal{H}=H^2$, the Hardy space. Then $K(z,w)=1/(1-z\overline{w})$, so $\frac{1}{K}(T,T^*)=I-TT^*$, and using the standard orthonormal basis $\psi_k(z)=z^k$ we get $f_m(T,T^*)=T^mT^{*m}$. Further, the operator M_z on the tensor product $H^2\otimes H$ is just the forward shift of infinite multiplicity (its wandering subspace is $\mathbf{1}\otimes H$). Thus we recover the result of Nagy-Foias mentioned in the beginning. $\square$

Other results covered by the last Theorem include the regular dilations of commuting n-tuples of operators (for $\mathcal{H}=$ the Hardy space on the polydisc $\mathbb{D}^n$), the k-hypercontractions of Agler [A2] (for $\mathcal{H}$ the Bergman space on $\mathbb{D}$ with respect to the weight $(1-|z|^2)^{k-2}$), and the "spherical hypercontractions" of Müller and Vasilescu [MV] (for $\mathcal{H}$ a certain weighted Bergman space on the unit ball of $\mathbb{C}^n$).

Sketch of the proof. (a) $\Longrightarrow$ (b). Denote $D_T=\frac{1}{K}(T,T^*)^{1/2}$. Define an operator $V:H\to\mathcal{H}\otimes H$ by

$$Vh=\sum_k\psi_k(z)\otimes D_T\psi_k(T)^*h. \tag{*}$$

We claim that V is well-defined (i.e. the sum converges) and is, in fact, an isometry satisfying $VT^*=M_z^*V$. This clearly establishes (b).

To see that V is well-defined and an isometry, observe that for any $j<m$ and $h\in H$

$$\begin{aligned}\Big\|\sum_{j\le k<m}\psi_k(z)\otimes D_T\psi_k(T)^*h\Big\|^2&=\sum_{j\le k<m}\|D_T\psi_k(T)^*h\|^2\\&=\sum_{j\le k<m}\langle\psi_k(T)D_T^2\psi_k(T)^*h,h\rangle\\&=\langle(f_j-f_m)(T,T^*)h,h\rangle.\end{aligned}$$

As $\langle f_m(T,T^*)h,h\rangle\to 0$ by hypothesis, it follows that the partial sums of the right-hand side of (*) form a Cauchy sequence, and letting $j=0$ and $m\to\infty$ shows that $\|Vh\|^2=\langle f_0(T,T^*)h,h\rangle=\|h\|^2$, i.e. V is an isometry.

To prove that $VT^* = M_z^* V$, observe that $\forall h, h' \in H$ and any k

$$\langle Vh, \psi_k \otimes h' \rangle = \langle D_T \psi_k(T)^* h, h' \rangle = \langle h, \psi_k(T) D_T h' \rangle,$$

so, by virtue of our choice of the basis ψ_k,

$$\langle Vh, f \otimes h' \rangle = \langle h, f(T) D_T h' \rangle \tag{$\dagger$}$$

for any polynomial f. Thus

$$\begin{aligned}\langle VT_j^* h, \psi_k \otimes h' \rangle &= \langle T_j^* h, \psi_k(T) D_T h' \rangle = \langle h, T_j \psi_k(T) D_T h' \rangle = \\ &= \langle h, (z_j \psi_k)(T) D_T h' \rangle = \langle Vh, z_j \psi_k \otimes h' \rangle = \langle M_{z_j}^* Vh, \psi_k \otimes h' \rangle,\end{aligned}$$

and the assertion follows.

(b) $\Longrightarrow$ (a). Let $U : H \to \mathcal{H} \otimes H$ be an isometry such that $UT^* = M_z^* U$. Then a simple calculation shows that for any polynomial $p(z, \overline{w})$ in z and $\overline{w}$,

$$p(T, T^*) = U^* p(M_z, M_z^*) U = U^* (p(Z, Z^*) \otimes I) U$$

(here, as before, Z stands for the operator tuple of multiplication by the coordinate functions on $\mathcal{H}$). Thus it suffices to show that $\frac{1}{K}(Z, Z^*) \geq 0$ and $\langle f_m(Z, Z^*) h, h \rangle \to 0 \ \forall h \in H$.

To see the former, recall that for any $w \in \Omega$

$$Z_j^* K_w = \overline{w}_j K_w$$

where $K_w(z) := K(z, w)$. It follows that for any polynomial $p(z, \overline{w})$ and $x, y \in \Omega$,

$$\langle p(Z, Z^*) K_y, K_x \rangle = p(x, \overline{y}) \langle K_y, K_x \rangle = p(x, \overline{y}) K(x, y),$$

so, in particular, $\langle \frac{1}{K}(Z, Z^*) K_y, K_x \rangle = 1 \ \forall x, y \in \Omega$. As also $\langle K_y, \mathbf{1} \rangle \langle \mathbf{1}, K_x \rangle = \overline{\mathbf{1}(y)} \mathbf{1}(x) = 1 \ \forall x, y \in \Omega$, it follows that

$$\frac{1}{K}(Z, Z^*) = \langle \cdot, \mathbf{1} \rangle \mathbf{1} \tag{**}$$

which is a positive operator.

For the latter assertion, observe that

$$\begin{aligned}(f_0 - f_m)(Z, Z^*) h &= \sum_{0 \leq k < m} \psi_k(Z) \tfrac{1}{K}(Z, Z^*) \psi_k(Z)^* h \\ &= \sum_{0 \leq k < m} \psi_k(Z) (\langle \cdot, \mathbf{1} \rangle \mathbf{1}) \psi_k(Z)^* h \qquad \text{by (**)} \\ &= \sum_{0 \leq k < m} \langle \psi_k(Z)^* h, \mathbf{1} \rangle \psi_k \\ &= \sum_{0 \leq k < m} \langle h, \psi_k \rangle \psi_k,\end{aligned}$$

and as ψ_k is an orthonormal basis and $f_0 \equiv 1$, it follows that

$$(f_0 - f_m)(Z, Z^*)h = h - f_m(Z, Z^*)h \to h \qquad \text{as } m \to \infty,$$

i.e. even $f_m(Z, Z^*)h \to 0\ \forall h \in H$. This completes the proof. $\square$

We remark that in view of the boundedness of V, the formula (†) defines a (non-multiplicative) "functional calculus" $g \mapsto g(T)$ for functions g on Ω of the form $g(z) = K(z, z)^{-1/2} f(z)$, $f \in \mathcal{H}$ (defining $g(T)h\,(=$"$f(T)D_T h$"$) := V^*(f \otimes h)$ one has $\|g(T)\| \leq \|V\| \|f\|$).

An example of function spaces $\mathcal{H}$ satisfying the axioms (1) – (4) are, for instance, various weighted Bergman and Hardy spaces on bounded symmetric domains in $\mathbb{C}^n$ (matrix balls etc.).

Under the additional hypothesis that the Taylor spectrum $\sigma(T) \subset \Omega$, it turns out that the condition $\langle f_m(T, T^*)h, h\rangle \to 0$ can be omitted, and the axioms (3) and (4) for $\mathcal{H}$ replaced by the weaker requirement that $K(z, w) \neq 0$ on $\Omega \times \Omega$. For this and further details we refer to the joint work [AEM].

References

[A1] J. Agler, *The Arveson extension theorem and coanalytic models*, Integral Equations and Operator Theory **5** (1982), 608–631.

[A2] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory **13** (1985), 201–217.

[MV] V. Müller, F.-H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. **117** (1993), 979–989.

[AEM] C. Ambrozie, M. Engliš, V. Müller, *Operator tuples and analytic models over general domains in* $\mathbb{C}^n$, preprint (1999).