A MOEBIUS INVARIANT SPACE OF H-HARMONIC FUNCTIONS ON THE BALL

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Abstract. We describe a Dirichlet-type space of H-harmonic functions, i.e. functions annihilated by the hyperbolic Laplacian on the unit ball of the real n-space, as the analytic continuation (in the spirit of Rossi and Vergne) of the corresponding weighted Bergman spaces. Characterizations in terms of derivatives are given, and the associated semi-inner product is shown to be Moebius invariant. We also give a formula for the corresponding reproducing kernel. Our results solve an open problem addressed by M. Stoll in his book "Harmonic and subharmonic function theory on the hyperbolic ball" (Cambridge University Press, 2016).

1. Introduction

Let B^n be the unit ball in the real n-space $\mathbf{R}^n$, $n > 2$, equipped with the usual hyperbolic metric $2(1-|x|^2)^{-1}\,|dx|$. The associated hyperbolic Laplace operator

$$\Delta_h f(x) := (1-|x|^2)[(1-|x|^2)\Delta f(x) + 2(n-2)\langle x, \nabla f(x)\rangle]$$

is invariant under the Moebius transformations of B^n. Functions on B^n annihilated by Δ_h are called hyperbolic-harmonic, or H-harmonic for short. The *weighted H-harmonic Bergman space*

$$\mathcal{H}_s(B^n) := \{f \in L^2(B^n, d\rho_s) : \ f \text{ is } H\text{-harmonic on } B^n\}$$

consists of all H-harmonic functions on B^n square-integrable with respect to the measure

$$d\rho_s(x) := \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}(1-|x|^2)^s\,dx, \qquad s > -1,$$

where dx denotes the Lebesgue volume on $\mathbf{R}^n$. The restriction on s ensures that these spaces are nontrivial, and the factor $\frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}$ makes $d\rho_s$ a probability measure, so that $\|\mathbf{1}\| = 1$. By Green's formula, H-harmonic functions possess the *invariant mean-value property*: namely, if $\Delta_h f = 0$ and $z \in B^n$, then $f(z)$ equals the mean value, with respect to the Moebius-invariant measure $d\tau(x) = (1-|x|^2)^{-n}\,dx$, over any Moebius ball in B^n centered at x. It follows by a standard argument that the point evaluations $f \mapsto f(x)$ at any $x \in B^n$ are continuous linear functionals on each $\mathcal{H}_s$, $s > -1$, and therefore there exists a reproducing kernel for

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$\mathcal{H}_s$ (the *weighted H-harmonic Bergman kernel*), namely a function $K_s(x,y)$ on $B^n \times B^n$, H-harmonic in both variables and such that

$$f(x) = \int_{B^n} f(y) K_s(x,y)\, d\rho_s(y) \qquad \forall x \in B^n, \forall f \in \mathcal{H}_s.$$

For the analogous weighted Bergman spaces of *holomorphic*, rather than H-harmonic, functions on the unit ball $\mathbf{B}^n \cong B^{2n}$ of $\mathbf{C}^n \cong \mathbf{R}^{2n}$,

$$\mathcal{A}_s := \{f \in L^2(\mathbf{B}^n, d\rho_s) : \ f \text{ is holomorphic on } \mathbf{B}^n\},$$

the reproducing kernels are given by the simple formula

$$K_s^{\mathrm{hol}}(x,y) = (1 - \langle x,y\rangle)^{-n-1}.$$

It is a remarkable fact — which prevails in the much more general context of bounded symmetric domains, constituting the "analytic continuation" of the principal series representations of certain semisimple Lie groups, cf. Rossi and Vergne [VR] — that these weighted Bergman kernels $K_s^{\mathrm{hol}}(x,y)$, $s > -1$, continue to be positive definite kernels in the sense of Aronszajn [Aro] for all $s \ge -n-1$, yielding thus an "analytic continuation" of the spaces $\mathcal{A}_s$. (One calls the interval $-n-1 \le s < +\infty$ the *Wallach set* of $\mathbf{B}^n$.) For $s > -n-1$, these spaces $\mathcal{A}_s$ turn out to be Besov-type spaces of analytic functions; for $s = -n-1$, the kernel $K_{-n-1}^{\mathrm{hol}}(x,y)$ becomes constant one, and the corresponding reproducing kernel Hilbert space thus reduces just to the constants. However, a much more interesting space arises as the "residue" of $\mathcal{A}_s$ at $s = -n-1$: namely, the limit

$$\lim_{s \searrow -n-1} \frac{K_s^{\mathrm{hol}}(x,y) - 1}{s+n+1} = \log \frac{1}{1 - \langle x,y\rangle}$$

is a positive definite kernel on $\mathbf{B}^n \times \mathbf{B}^n$, and the associated reproducing kernel Hilbert space is nothing else but the familiar *Dirichlet space* on $\mathbf{B}^n$. Furthermore, the semi-inner product in this space turns out to be *Moebius invariant*, in the sense that

$$\langle f, g\rangle = \langle f \circ \phi, g \circ \phi\rangle$$

for any biholomorphic self-map ϕ of $\mathbf{B}^n$. See e.g. Chapter 6.4 in Zhu [Zh].

It has recently been shown by two of the current authors that a similar situation as described in the previous paragraph prevails also for spaces of M-harmonic functions on $\mathbf{B}^n$ (i.e. functions annihilated by the Poincaré Laplacian on $\mathbf{B}^n$ invariant under the action of the biholomorphic self-maps of $\mathbf{B}^n$), as well as for spaces of harmonic and pluriharmonic functions. The aim of the current paper is to treat the case of H-harmonic functions on B^n.

Quite surprisingly (at least for the current authors), it turns out that the situation in the H-harmonic case is strikingly different from all the cases above, in that the "Wallach set" of those real s for which $K_s(x,y)$ continue to be positive definite kernels, does not have the form of an interval, with an appropriate "Dirichlet space" arising as the limit (or, rather, residue) at the lower endpoint. Instead, K_s stop to be positive definite some way below $s = -1$, only to resurface into positive definite kernels at the point $s = -n$. In particular, the "H-harmonic Wallach set" of B^n is thus disconnected. Furthermore, although we do not have any closed formula for the corresponding reproducing kernel, we are able to show that the associated semi-inner product is, miraculously, indeed invariant under composition with Moebius maps of B^n. In particular, this gives an answer to Question 2 on page 180 of the book of M. Stoll [St], in exhibiting a Moebius-invriant Hilbert space of H-harmonic functions.

Finally, just as the ordinary (i.e. holomorphic) Dirichlet space on the disc consists of all holomorphic functions whose first derivative is square-integrable over the disc, we give characterizations of our H-harmonic Dirichlet space in terms of square-integrability of derivatives of appropriate order. This is very similar in spirit to the results of Grellier and Jaming [Ja] [GJ].

The details of the construction of our "H-harmonic Dirichlet space" are presented in Section 3, after recalling the necessary background material in Section 2. The characterization in terms of derivatives is given in Section 4, and the invariance of the semi-inner product is proved in Section 5. In Section 6, we identify the reproducing kernel of our Dirichlet space. The last section, Section 7, contains some final remarks and comments.

Throughout the paper, the notation

$$A \asymp B$$

means that

$$cA \leq B \leq \frac{1}{c} A$$

for some $0 < c \leq 1$ independent of the variables in question. If only the first of these inequalities is fulfilled, we write $A \lesssim B$; and $A \sim B$ means that A/B tends to 1. The partial derivatives $\partial/\partial x_j$ are commonly abbreviated just to ∂_j, and similarly ∂_r stands for $\partial/\partial r$ and so forth; and for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ a multi-index, ∂^α stands for $\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. To make typesetting a little neater, the shorthand

$$\Gamma\binom{a_1, a_2, \dots, a_k}{b_1, b_2, \dots, b_m} := \frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_k)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_m)}$$

is often employed throughout the paper. Finally, $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ denote the sets of all integers, all nonnegative integers, all real and all complex numbers, respectively.

2. Notation and preliminaries

Let B^n be the unit ball in $\mathbf{R}^n$, $n > 2$. The orthogonal transformations

$$x \longmapsto Ux, \qquad x \in \mathbf{R}^n,\ U \in O(n),$$

map both B^n and its boundary ∂B^n (the unit sphere) onto themselves, and so do the Moebius transformations

$$\phi_a(x) := \frac{a|x-a|^2 + (1-|a|^2)(a-x)}{[x,a]^2}$$

interchanging the origin $0 \in \mathbf{R}^n$ with some point $a \in B^n$; here $[x, a]$ is defined as

$$[x,a] := \sqrt{1 - 2\langle x, a\rangle + |x|^2|a|^2}.$$

The group generated by the ϕ_a, $a \in B^n$, and $O(n)$ via composition is called the Moebius group of B^n; it is actually true that any element of this group can be written just as $U\phi_a$ with some $U \in O(n)$ and $a \in B^n$. The hyperbolic Laplacian

$$\Delta_h f(x) := (1-|x|^2)[(1-|x|^2)\Delta f(x) + 2(n-2)\langle x, \nabla f(x)\rangle]$$

is Moebius invariant, i.e. commutes with the action of this group. Functions on B^n annihilated by Δ_h are called hyperbolic-harmonic, or H-harmonic for short. The *weighted H-harmonic Bergman space*

$$\mathcal{H}_s(B^n) := \{f \in L^2(B^n, d\rho_s) : \ f \text{ is } H\text{-harmonic on } B^n\}$$

consists of all H-harmonic functions on B^n square-integrable with respect to the measure

$$d\rho_s(x) := \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}(1-|x|^2)^s \, dx, \qquad s > -1, \tag{1}$$

where dx denotes the Lebesgue volume on $\mathbf{R}^n$. The restriction on s ensures that these spaces are nontrivial, and the factor $\frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)}$ makes $d\rho_s$ a probability measure, so that $\|\mathbf{1}\| = 1$.

For an integer $m \geq 0$, let $\mathcal{H}^m$ be the space of restrictions to the unit sphere ∂B^n of harmonic polynomials on $\mathbf{R}^n$ homogeneous of degree m. We refer to [ABR], especially Chapter 5, for the Peter-Weyl decomposition

$$L^2(\partial B^n, d\sigma) = \bigoplus_{m=0}^{\infty} \mathcal{H}^m \tag{2}$$

under the action of the orthogonal group $O(n)$ of rotations of $\mathbf{R}^n$. Here and throughout $d\sigma$ stands for the normalized surface measure on the unit sphere ∂B^n. Performing such a decomposition on each sphere $|z| \equiv$ const. leads to the analogous Peter-Weyl decomposition

$$\mathcal{H}_s = \bigoplus_m \mathbf{H}^m,$$

of the weighted H-harmonic Bergman spaces, where $\mathbf{H}^m$ is the space of "solid harmonics"

$$\mathbf{H}^m = \{f \in C(\overline{B^n}) : \ f \text{ is } H\text{-harmonic on } B^n \text{ and } f|_{\partial B^n} \in \mathcal{H}^m\},$$

and the norm of $f = \sum_m f_m$, $f_m \in \mathbf{H}^m$, is given by

$$\|f\|_s^2 = \sum_{m=0}^{\infty} I_m(s) \|f_m\|^2_{\partial B^n}, \tag{3}$$

with the coefficients $I_m(s)$ given by the explicit formula

$$I_m(s) := \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \int_0^1 t^{m+\frac{n}{2}-1}(1-t)^s S_m(t)^2 \, dt, \tag{4}$$

where

$$S_m(t) := \frac{(n-1)_m}{(\frac{n}{2})_m} {}_2F_1\left(\begin{matrix} m, 1-\frac{n}{2} \\ m+\frac{n}{2} \end{matrix} \middle| t\right) \tag{5}$$

with the Gauss hypergeometric function ${}_2F_1$ and the Pochhammer symbol $(x)_m := x(x+1)\dots(x+m-1)$. Also, any $f_m \in \mathbf{H}^m$ is necessarily of the form

$$f_m(r\zeta) = S_m(r^2) r^m f_m(\zeta), \quad f_m|\partial B^n \in \mathcal{H}^m, \qquad 0 \leq r \leq 1, \zeta \in \partial B^n. \tag{6}$$

It follows that the space $\mathcal{H}_s$ has reproducing kernel given by

$$K_s(x,y) = \sum_{m=0}^{\infty} \frac{S_m(|x|^2) S_m(|y|^2)}{I_m(s)} Z_m(x,y), \tag{7}$$

where $Z_m(x,y)$, the zonal harmonic of degree m, is the reproducing kernel of $\mathcal{H}^m$ extended by homogeneity to all $x, y \in B^n$; see Chapter 8 in [ABR].

The reader is referred to [St] for a detailed exposition of the facts above, with convenient overview and further developments in [St2] and [Ur]. Note however that, for ease of notation, our $S_m(t)$ is $S_m(\sqrt{t})$ in the notation of these references.

3. Analytic continuation

Throughout this paper, we assume that $n > 2$. (For $n = 2$, H-harmonic functions coincide with ordinary harmonic ones, and most things work out differently and, in fact, become much simpler; cf. Section 6 in [EY2].)

Proposition 1. *The coefficient functions $I_m(s)$, $m > 0$, extend to meromorphic functions of s on the entire complex plane, with a simple pole at $s = -n$, with residue $(n-1)_m/\Gamma(m)$ for $n > 2$ even, and $2(n-1)_m/\Gamma(m)$ for $n > 2$ odd.*

For the proof, we will need the following simple lemma.

Lemma 2. *Let $m \in \mathbf{N}$, $n > 1$, $s > -1$. Then for all $\alpha \geq 0$:*

$$\int_0^1 t^{m+\frac{n}{2}-1}(1-t)^{s+\alpha} S_m(t)\,dt = \frac{(n-1)_m \Gamma(s+1+\alpha)\Gamma(n+s+\alpha)\Gamma(\frac{n}{2})}{\Gamma(m+n+s+\alpha)\Gamma(\frac{n}{2}+s+1+\alpha)}. \tag{8}$$

Proof. From definition of S_m we have

$$\begin{aligned}\int_0^1 t^{m+\frac{n}{2}-1}(1-t)^{s+\alpha} S_m(t)\,dt &= \frac{(n-1)_m}{(\frac{n}{2})_m}\int_0^1 t^{m+\frac{n}{2}-1}(1-t)^{s+\alpha}{}_2F_1\left(\begin{matrix}m, 1-\frac{n}{2}\\ m+\frac{n}{2}\end{matrix}\middle| t\right)dt\\ &= \frac{(n-1)_m}{(\frac{n}{2})_m}\int_0^1 t^{m+\frac{n}{2}-1}(1-t)^{s+\alpha}\sum_{k=0}^\infty \frac{(m)_k(1-\frac{n}{2})_k}{(m+\frac{n}{2})_k k!}t^k\,dt.\end{aligned}$$

The series converges uniformly for all $t \in [0,1]$ when $n > 1$ by a simple ratio test. Therefore we can integrate term by term to get

$$\begin{aligned}&= \frac{(n-1)_m}{(\frac{n}{2})_m}\sum_{k=0}^\infty \frac{(m)_k(1-\frac{n}{2})_k}{(m+\frac{n}{2})_k k!}\frac{\Gamma(m+\frac{n}{2}+k)\Gamma(s+\alpha+1)}{\Gamma(m+\frac{n}{2}+s+\alpha+1+k)}\\ &= \frac{(n-1)_m\Gamma(m+\frac{n}{2})\Gamma(s+\alpha+1)}{(\frac{n}{2})_m\Gamma(m+\frac{n}{2}+s+\alpha+1)}{}_2F_1\left(\begin{matrix}m, 1-\frac{n}{2}\\ m+\frac{n}{2}+s+\alpha+1\end{matrix}\middle|1\right)\\ &= \frac{(n-1)_m\Gamma(s+1+\alpha)\Gamma(n+s+\alpha)\Gamma(\frac{n}{2})}{\Gamma(m+n+s+\alpha)\Gamma(\frac{n}{2}+s+1+\alpha)},\end{aligned}$$

by Gauss's summation formula [BE, §2.1 (14)]. □

Proof of Proposition 1. Using the formula [BE, §2.10 (12)] on $S_m(t)$ we obtain

$$\begin{aligned}S_m(t) &= \sum_{k=0}^{n-2}\frac{(m)_k(1-\frac{n}{2})_k}{k!(2-n)_k}(1-t)^k\\ &\quad+\frac{(-1)^n}{\Gamma(m)\Gamma(1-\frac{n}{2})\Gamma(n-1)}\sum_{k=0}^\infty\frac{\Gamma(\frac{n}{2}+k)\Gamma(m+n-1+k)}{\Gamma(k+1)\Gamma(k+n)}(1-t)^{k+n-1}\\ &\quad\times\Big(\log(1-t)-\psi(k+1)-\psi(k+n)+\psi(\frac{n}{2}+k)+\psi(m+n+k-1)\Big).\end{aligned}$$

This can be rewritten as

(9)
$$S_m(t)=\sum_{k=0}^{n-2}\frac{(m)_k(1-\frac{n}{2})_k}{k!(2-n)_k}(1-t)^k \\ +\frac{(-1)^n}{\Gamma(m)\Gamma(1-\frac{n}{2})\Gamma(n-1)}\sum_{k=0}^{\infty}[\epsilon]\frac{\Gamma(\frac{n}{2}+k+\epsilon)\Gamma(m+n-1+k+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(k+n+\epsilon)}(1-t)^{k+n-1+\epsilon},$$

where $[\epsilon]f(\epsilon):=f'(0)$.

Replacing one of the S_m in (4) by (9) and applying Lemma 2 term by term we obtain

$$I_m(s)=A(s)+B(s),$$

where

$$A(s)=\frac{\Gamma(n+s)(n-1)_m}{\Gamma(m+n+s)}\sum_{k=0}^{n-2}\frac{(m)_k(1-\frac{n}{2})_k(s+1)_k(n+s)_k}{k!(2-n)_k(\frac{n}{2}+s+1)_k(m+n+s)_k},$$

$$B(s)=\frac{(-1)^n(n-1)_m\Gamma(\frac{n}{2}+s+1)}{\Gamma(m)\Gamma(1-\frac{n}{2})\Gamma(n-1)\Gamma(s+1)} \\ \times\sum_{k=0}^{\infty}[\epsilon]\frac{\Gamma(\frac{n}{2}+k+\epsilon)\Gamma(m+n-1+k+\epsilon)\Gamma(s+n+k+\epsilon)\Gamma(2n+s-1+k+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(k+n+\epsilon)\Gamma(\frac{3n}{2}+s+k+\epsilon)\Gamma(m+2n+s-1+k+\epsilon)}.$$

It is easy to check that the k-th term of the series in $B(s)$ behaves like k^{-2} for large k, hence the series converges absolutely for all values of s and m (as long as we avoid the singularities of the Γ functions, of course).

Using the fact that

$$[\epsilon]\Gamma(a+\epsilon)=\Gamma(a)^2[\epsilon]\frac{1}{\Gamma(a-\epsilon)},$$

we can write $A(s)$, $B(s)$ in the more illuminating form

$$A(s)=\Gamma(n+s)(n-1)_m\Gamma(\frac{n}{2}+s+1)\sum_{k=0}^{n-2}\frac{(m)_k(1-\frac{n}{2})_k(s+1)_k(n+s)_k}{k!(2-n)_k\Gamma(\frac{n}{2}+s+1+k)\Gamma(m+n+s+k)},$$

$$B(s)=\frac{(-1)^n(n-1)_m\Gamma(\frac{n}{2}+s+1)\Gamma(s+n)\Gamma(2n+s-1)^2(s+1)_{n-1}}{\Gamma(m)\Gamma(1-\frac{n}{2})\Gamma(n-1)} \\ \times\sum_{k=0}^{\infty}[\epsilon]\frac{\Gamma(\frac{n}{2}+k+\epsilon)\Gamma(m+n-1+k+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(k+n+\epsilon)\Gamma(\frac{3n}{2}+s+k+\epsilon)\Gamma(m+2n+s-1+k+\epsilon)} \\ \times\frac{(s+n)_k^2(2n+s-1)_k^2}{\Gamma(s+n+k-\epsilon)\Gamma(2n+s+k-1-\epsilon)},$$

which reveals that both A and B are meromorphic. In fact,

$$A(s)=\Gamma(n+s)\Gamma(\frac{n}{2}+s+1)\times(\text{an entire function of } s),$$
$$B(s)=\Gamma(n+s)\Gamma(\frac{n}{2}+s+1)\Gamma(2n+s-1)^2\times(\text{an entire function of } s).$$

Thus the function $I_m(s)$ can have a pole only at $s=-n-j$ or $s=-1-n/2-j$ or $s=1-2n-j$, $j\in\mathbf{N}$.

Using the fact that

$$\lim_{s\to -n}(n+s)\Gamma(n+s)=1,$$

we can directly compute

$$\lim_{s\to -n}(s+n)A(s)=(n-1)_m\sum_{k=0}^{n-2}\frac{(m)_k(1-\frac{n}{2})_k(1-n)_k(0)_k}{k!(2-n)_k(1-\frac{n}{2})_k\Gamma(m+k)}=\frac{(n-1)_m}{\Gamma(m)}.$$

Since for n even $B(s)$ vanishes identically (due to the presence of $\Gamma(1-n/2)$ in the denominator) this proves the proposition in the case of $n>2$ even.

For $n>2$ odd, a similar computation yields

$$\begin{aligned}\lim_{s\to -n}(s+n)B(s)=&\frac{-(n-1)_m\Gamma(1-\frac{n}{2})\Gamma(n-1)^2\Gamma(n)}{\Gamma(m)\Gamma(1-\frac{n}{2})\Gamma(n-1)}\\&\times\sum_{k=0}^{\infty}[\epsilon]\frac{\Gamma(\frac{n}{2}+k+\epsilon)\Gamma(m+n-1+k+\epsilon)}{\Gamma(k+1+\epsilon)\Gamma(k+n+\epsilon)\Gamma(\frac{n}{2}+k+\epsilon)\Gamma(m+n-1+k+\epsilon)}\\&\times\frac{(0)_k^2(n-1)_k^2}{\Gamma(k-\epsilon)\Gamma(n+k-1-\epsilon)}.\end{aligned}$$

The resulting series has only a single non-zero term for $k=0$. And even for this term we must spend the derivative with respect to ϵ on the factor

$$\frac{1}{\Gamma(k-\epsilon)},$$

otherwise we will get zero. Since

$$\lim_{\epsilon\to 0}\frac{\Gamma'(-\epsilon)}{\Gamma(-\epsilon)^2}\to -1,$$

we thus obtain

$$\lim_{s\to -n}(s+n)B(s)=\frac{(n-1)_m\Gamma(n-1)\Gamma(n)}{\Gamma(m)}\frac{\Gamma(\frac{n}{2})\Gamma(m+n-1)}{\Gamma(1)\Gamma(n)\Gamma(\frac{n}{2})\Gamma(m+n-1)}\frac{1}{\Gamma(n-1)}=\frac{(n-1)_m}{\Gamma(m)}.$$

Combining both results we get

$$\lim_{s\to -n}(n+s)I_m(s)=\lim_{s\to -n}(n+s)(A(s)+B(s))=2\frac{(n-1)_m}{\Gamma(m)},$$

completing the proof for $n>2$ odd. □

It was shown in Theorem 3.1 in [Ur] that

$$I_m(s)\asymp (m+1)^{-s-1} \tag{10}$$

for any fixed $s>-1$. We will need a similar result for the somewhat more general integral

$$I_{m,k}(s):=\frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)}\int_0^1[(2t\partial_t)^k(t^{m/2}S_m(t))]^2\,t^{\frac{n}{2}-1}(1-t)^s\,dt, \tag{11}$$

which reduces to (4) for $k=0$.

Proposition 3. *For a fixed $s>-1$ and $k\in\{0,1,\dots,n-2\}$,*

$$I_{m,k}(s)\asymp (m+1)^{-s-1+2k}. \tag{12}$$

Actually, we even have the following complete asymptotic expansion, of which (12) is just the leading order term.

Proposition 4. *Assume that $s > -1$ and $k < n-1$. Then there exist constants A_j depending only on s, k and n such that*

$$I_{m,k}(s) \approx \sum_{j=0}^{\infty} \frac{A_j}{m^{s+1-2k+j}} \qquad \text{as } m \to \infty,$$

with $A_0 > 0$.

For the proof we are going to need several lemmas.

Lemma 5. *Let*

$$c_{j,k}(m) := (m)_j \frac{\Delta_-^j}{j!}(2y-m)^k|_{y=0},$$

where Δ_- is the backward difference operator, i.e.

$$\Delta_- f(y) = f(y) - f(y-1).$$

Then

$$I_{m,k}(s) = \frac{\Gamma(1+\frac{n}{2}+s)(n-1)_m^2}{\Gamma(\frac{n}{2})\Gamma(s+1)(\frac{n}{2})_m^2} \sum_{j,l=0}^{k} c_{j,k}(m) c_{l,k}(m) \tilde{I}_{j,l},$$

where

$$\tilde{I}_{j,l} := \int_0^1 t^{m+\frac{n}{2}-1}(1-t)^s {}_2F_1\left(\begin{matrix} m+j, 1-\frac{n}{2} \\ m+\frac{n}{2} \end{matrix}\middle| t\right) {}_2F_1\left(\begin{matrix} m+l, 1-\frac{n}{2} \\ m+\frac{n}{2} \end{matrix}\middle| t\right) dt.$$

Proof. The proof is based on the fact that for all x:

$$(2x)^k = \sum_{j=0}^{k} \frac{c_{j,k}(m)}{(m)_j}\left(x+\frac{m}{2}\right)_j, \tag{13}$$

and the fact that

$$\left(t\partial_t + \frac{m}{2}\right)_j t^{\frac{m}{2}} {}_2F_1\left(\begin{matrix} m, 1-\frac{n}{2} \\ m+\frac{n}{2} \end{matrix}\middle| t\right) = (m)_j t^{\frac{m}{2}} {}_2F_1\left(\begin{matrix} m+j, 1-\frac{n}{2} \\ m+\frac{n}{2} \end{matrix}\middle| t\right).$$

□

Remark 6. Replacing x by $mx + \frac{m}{2}$ in (13), dividing by m^k and letting $m \to \infty$ shows that

$$c_{j,k}(m) \sim m^k \binom{k}{j} (-1)^{k-j} 2^j \qquad \text{as } m \to \infty. \tag{14}$$

With a little more effort, it is possible to obtain in the same way the complete asymptotic expansion of $c_{j,k}(m)$ in decreasing powers of m. □

Lemma 7. *Let $\sigma_1 := c+d-a-b$, $\sigma_2 := \gamma - \alpha - \beta$, $\sigma := \sigma_1 + \sigma_2$. Then*

(15)

$$\int_0^1 t^{c-1}(1-t)^{d-1} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| t\right) {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}\middle| t\right) dt = \Gamma\left(\begin{matrix} c, \gamma, d, \sigma_1, \sigma \\ \gamma-\beta, \sigma_1+a, \sigma_1+b, \sigma+\beta \end{matrix}\right)$$
$$\times \sum_{k=0}^{\infty} \frac{(\beta)_k(\sigma_1)_k(\sigma_1+a-\alpha)_k(d)_k}{(\sigma+\beta)_k(\sigma_1+a)_k(\sigma_1+b)_k k!} {}_3F_2\left(\begin{matrix} \beta+k, \sigma_1+k, c-a \\ \sigma+\beta+k, \sigma_1+b+k \end{matrix}\middle| 1\right),$$

whenever both sides makes sense.

Remark 8. A sufficient condition for the integral in (15) to converge is $c > 0$, $\sigma_1 > d > 0$, $\sigma_2 > 0$.

A sufficient condition for the right-hand side to converge is $\gamma > \beta$, $\sigma_2 + d > 0$ (except for pathological values of parameters).

The condition $\sigma_2+d > 0$ is needed for the value ${}_3F_2(1)$ to exist, and $\gamma > \beta$ ensures that the whole series converges. This can be seen employing the well known formula [Olv, 16.4.11]

(16)
$$ {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ c_1, c_2 \end{matrix}\middle| 1\right) = \Gamma\left(\begin{matrix} c_2, c_1 + c_2 - a_1 - a_2 - a_3 \\ c_2 + c_1 - a_1 - a_2, c_2 - a_3 \end{matrix}\right) {}_3F_2\left(\begin{matrix} a_3, c_1 - a_1, c_1 - a_2 \\ c_1, c_1 + c_2 - a_1 - a_2 \end{matrix}\middle| 1\right). $$

Hence in our case

$$ {}_3F_2\left(\begin{matrix} \beta + k, \sigma_1 + k, c - a \\ \sigma + \beta + k, \sigma_1 + b + k \end{matrix}\middle| 1\right) = \Gamma\left(\begin{matrix} \sigma_1 + b + k, \sigma_2 + d \\ \sigma + b, d + k \end{matrix}\right) {}_3F_2\left(\begin{matrix} \sigma, \sigma_2 + \beta, c - a \\ \sigma + \beta + k, \sigma + b \end{matrix}\middle| 1\right). $$

Therefore

$$ {}_3F_2\left(\begin{matrix} \beta + k, \sigma_1 + k, c - a \\ \sigma + \beta + k, \sigma_1 + b + k \end{matrix}\middle| 1\right) \sim \frac{\Gamma(\sigma_2 + d)}{\Gamma(\sigma + b)} k^{c-a} \qquad \text{as } k \to \infty, $$

by the known large parameter asymptotics of ${}_3F_2$. See [Olv, 16.11.10]. Taking into account the behavior of all the Pochhammer symbols in the series, we obtain that the k-th term behaves like $k^{\beta-\gamma-1}$. Thus, indeed, $\gamma > \beta$ is sufficient for convergence. □

Proof. Denote the right hand side of (15) by I. First we represent the ${}_3F_2$ function in I by a double integral:

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} \beta + k, \sigma_1 + k, c - a \\ \sigma + \beta + k, \sigma_1 + b + k \end{matrix}\middle| 1\right) &= \frac{(\sigma + \beta)_k (\sigma_1 + b)_k}{(\beta)_k (\sigma_1)_k} \Gamma\left(\begin{matrix} \sigma + \beta, \sigma_1 + b \\ \beta, \sigma_1, \sigma, b \end{matrix}\right) \\ &\times \int_0^1 \int_0^1 x^{\beta+k-1} (1 - x)^{\sigma-1} y^{\sigma_1+k-1} (1 - y)^{b-1} (1 - xy)^{a-c} \, dx \, dy, \end{aligned} $$

and then swap the order of integration and summation so that I becomes

$$ \begin{aligned} I = \Gamma&\left(\begin{matrix} c, \gamma, d \\ \gamma - \beta, \sigma_1 + a, \beta, b \end{matrix}\right) \\ &\times \int_0^1 \int_0^1 x^{\beta-1} (1 - x)^{\sigma-1} y^{\sigma_1-1} (1 - y)^{b-1} (1 - xy)^{a-c} {}_2F_1\left(\begin{matrix} \sigma_1 + a - \alpha, d \\ \sigma_1 + a \end{matrix}\middle| xy\right) dx \, dy. \end{aligned} $$

Next we represent the ${}_2F_1$ function by a single integral:

$$ {}_2F_1\left(\begin{matrix} \sigma_1 + a - \alpha, d \\ \sigma_1 + a \end{matrix}\middle| xy\right) = \Gamma\left(\begin{matrix} \sigma_1 + a \\ \alpha, \sigma_1 + a - \alpha \end{matrix}\right) \int_0^1 t^{\sigma_1+a-\alpha-1} (1 - t)^{\alpha-1} (1 - txy)^{-d} \, dt, $$

yielding

$$ \begin{aligned} I = \Gamma&\left(\begin{matrix} c, \gamma, d \\ \gamma - \beta, \beta, b, \alpha, \sigma_1 + a - \alpha \end{matrix}\right) \\ &\times \int_0^1 \int_0^1 \int_0^1 x^{\beta-1} (1 - x)^{\sigma-1} y^{\sigma_1-1} (1 - y)^{b-1} t^{\sigma_1+a-\alpha-1} (1 - t)^{\alpha-1} (1 - xy)^{a-c} (1 - txy)^{-d} \, dt \, dx \, dy. \end{aligned} $$

Integrating with respect to the y variable produces

$$ I = \Gamma\left(\begin{matrix} c, \gamma, d, \sigma_1 \\ \gamma - \beta, \beta, \sigma_1 + b, \alpha, \sigma_1 + a - \alpha \end{matrix}\right) $$

$$\times \int_0^1 \int_0^1 x^{\beta-1}(1-x)^{\sigma-1} t^{\sigma_1+a-\alpha-1}(1-t)^{\alpha-1} F_1\Big(\begin{matrix}\sigma_1; c-a, d\\ \sigma_1+b\end{matrix}\Big| x, tx\Big)\, dt\, dx,$$

where F_1 is the first Appell hypergeometric function. The well known transformation rule [Olv, 16.16.1]

$$F_1\Big(\begin{matrix}a; b_1, b_2\\ b_1+b_2\end{matrix}\Big| x, y\Big) = (1-x)^{-a} {}_2F_1\Big(\begin{matrix}a, b_2\\ b_1+b_2\end{matrix}\Big| \frac{x-y}{x-1}\Big)$$

implies

$$F_1\Big(\begin{matrix}\sigma_1; c-a, d\\ \sigma_1+b\end{matrix}\Big| x, tx\Big) = (1-x)^{-\sigma_1} {}_2F_1\Big(\begin{matrix}\sigma_1, d\\ \sigma_1+b\end{matrix}\Big| \frac{x}{x-1}(1-t)\Big).$$

Substituting this into the last formula for I we get

$$I = \Gamma\Big(\begin{matrix}c, \gamma, d, \sigma_1\\ \gamma-\beta, \beta, \sigma_1+b, \alpha, \sigma_1+a-\alpha\end{matrix}\Big)$$
$$\times \int_0^1 \int_0^1 x^{\beta-1}(1-x)^{\sigma_2-1} t^{\sigma_1+a-\alpha-1}(1-t)^{\alpha-1} {}_2F_1\Big(\begin{matrix}\sigma_1, d\\ \sigma_1+b\end{matrix}\Big| \frac{x}{x-1}(1-t)\Big)\, dt\, dx.$$

This can be integrated with respect to t:

$$I = \Gamma\Big(\begin{matrix}c, \gamma, d, \sigma_1\\ \gamma-\beta, \beta, \sigma_1+a, \sigma_1+b\end{matrix}\Big) \int_0^1 x^{\beta-1}(1-x)^{\sigma_2-1} {}_3F_2\Big(\begin{matrix}\sigma_1, d, \alpha\\ \sigma_1+b, \sigma_1+a\end{matrix}\Big| \frac{x}{x-1}\Big)\, dx.$$

Now we employ Lemma 2.10 in [Ur] which asserts that

$$\int_0^1 t^{c-1}(1-t)^{d-1} {}_2F_1\Big(\begin{matrix}a, b\\ c\end{matrix}\Big| t\Big)(1-tx)^{-\alpha}\, dt = \Gamma\Big(\begin{matrix}c, d, \sigma_1\\ \sigma_1+a, \sigma_1+b\end{matrix}\Big)(1-x)^{-\alpha}$$
$$\times {}_3F_2\Big(\begin{matrix}\sigma_1, d, \alpha\\ \sigma_1+b, \sigma_1+a\end{matrix}\Big| \frac{x}{x-1}\Big).$$

Using this we get

$$I = \Gamma\Big(\begin{matrix}\gamma\\ \gamma-\beta, \beta\end{matrix}\Big) \int_0^1 \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1} t^{c-1}(1-t)^{d-1} {}_2F_1\Big(\begin{matrix}a, b\\ c\end{matrix}\Big| t\Big)(1-tx)^{-\alpha}\, dt\, dx.$$

A final integration with respect to x gives us what we want:

$$I = \int_0^1 t^{c-1}(1-t)^{d-1} {}_2F_1\Big(\begin{matrix}a, b\\ c\end{matrix}\Big| t\Big) {}_2F_1\Big(\begin{matrix}\alpha, \beta\\ \gamma\end{matrix}\Big| t\Big)\, dt.$$

This proves (15) for values of a, b, c, d, α, β, γ for which all the operations above make sense and all integrals and series converge uniformly. But since both sides of (15) are meromorphic functions (of all parameters) in their respective domains, we can extend the validity of this argument by analytic continuation. □

Corollary 9. *For $n+s>l$ and $m+n>1$,*

$$\frac{(n-1)_m^2}{(\frac{n}{2})_m^2} \tilde{I}_{j,l} = \Gamma\Big(\begin{matrix}m+n-1\\ m+n+s\end{matrix}\Big) \Gamma\Big(\begin{matrix}\frac{n}{2}, \frac{n}{2}, s+1, n+s-j, 2n+s-1-j-l\\ n-1, n-1, \frac{n}{2}+s+1-j, \frac{3n}{2}+s-j-l\end{matrix}\Big) \tag{17}$$
$$\times \sum_{k=0}^{\infty} \frac{(1-\frac{n}{2})_k (n+s-j)_k (n+s-l)_k (s+1)_k}{(\frac{3n}{2}+s-j-l)_k (m+n+s)_k (\frac{n}{2}+s+1-j)_k k!}$$
$$\times {}_3F_2\Big(\begin{matrix}1-\frac{n}{2}+k, n+s-j+k, \frac{n}{2}-j\\ \frac{3n}{2}+s-j-l+k, \frac{n}{2}+s+1-j+k\end{matrix}\Big| 1\Big).$$

Proof. Apply the last lemma to the integral defining $\tilde{I}_{j,l}$. □

Corollary 10. *For $n+s>l$, $n+s>j$ and $s>-1$, as $m\to\infty$:*

$$\frac{(n-1)_m^2}{(\frac{n}{2})_m^2}\tilde{I}_{j,l}\sim m^{-s-1}\Gamma\binom{\frac{n}{2},\frac{n}{2},s+1,n+s-j,n+s-l}{n-1,n-1,\frac{n}{2}+s+1-j,\frac{n}{2}+s+1-l}\times {}_3F_2\left(\begin{matrix}1-\frac{n}{2},s+1,1-\frac{n}{2}\\ \frac{n}{2}+s+1-j,\frac{n}{2}+s+1-l\end{matrix}\middle|1\right).$$

Proof. Apply the formula (16) to the first term in (17). □

Corollary 11. *For $s>-1$ and $k<n-1$, as $m\to\infty$:*

$$I_{m,k}(s)\sim m^{2k-s-1}\Gamma\binom{\frac{n}{2},1+\frac{n}{2}+s,n+s-k,n+s-k}{n-1,n-1,\frac{n}{2}+1+s-k,\frac{n}{2}+1+s-k}\times\sum_{j=0}^{\infty}\frac{(1-\frac{n}{2})_j^2(s+1)_j}{(1+\frac{n}{2}+s-k)_j^2j!}\,{}_2F_1\left(\begin{matrix}-k,1-\frac{n}{2}+j\\ 1+\frac{n}{2}+s-k+j\end{matrix}\middle|-1\right)^2. \tag{18}$$

Proof. Combining the previous corollary, the definition of $I_{m,k}(s)$ and the asymptotics (14) of the coefficients $c_{j,k}(m)$, $c_{l,k}(m)$ we get as $m\to\infty$:

$$I_{m,k}(s)\sim m^{2k-s-1}\Gamma\binom{\frac{n}{2},n+s,n+s}{n-1,n-1,\frac{n}{2}+1+s}\times\sum_{j,l=0}^{k}\frac{(-k)_j(-k)_l(-\frac{n}{2}-s)_j(-\frac{n}{2}-s)_l}{j!k!(1-n-s)_j(1-n-s)_l}2^{j+l}\,{}_3F_2\left(\begin{matrix}1-\frac{n}{2},s+1,1-\frac{n}{2}\\ \frac{n}{2}+s+1-j,\frac{n}{2}+s+1-l\end{matrix}\middle|1\right).$$

Note that the series for ${}_3F_2(1)$ converges for $s+2n-2k-1>0$ which is satisfied due to our assumptions $s>-1$, $k<n-1$.

Expanding the ${}_3F_2$ into series and using the fact that

$$\frac{(-\frac{n}{2}-s)_j}{(1+\frac{n}{2}+s-j)_r}=\frac{(-\frac{n}{2}-s-r)_j}{(1+\frac{n}{2}+s)_r}$$

for all r, we obtain

$$I_{m,k}(s)\sim m^{2k-s-1}\Gamma\binom{\frac{n}{2},n+s,n+s}{n-1,n-1,\frac{n}{2}+1+s}\times\sum_{r=0}^{\infty}\frac{(1-\frac{n}{2})_r^2(s+1)_r}{(1+\frac{n}{2}+s)_r^2r!}\,{}_2F_1\left(\begin{matrix}-k,-\frac{n}{2}-s-r\\ 1-n-s\end{matrix}\middle|2\right)^2. \tag{19}$$

We now use the transform [Olv, 15.8.6] to get

$${}_2F_1\left(\begin{matrix}-k,-\frac{n}{2}-s-r\\ 1-n-s\end{matrix}\middle|2\right)=(-2)^k\Gamma\binom{n+s-k,1+\frac{n}{2}+s}{n+s,1+\frac{n}{2}+s-k}\times\frac{(1+\frac{n}{2}+s)_r}{(1+\frac{n}{2}+s-k)_r}\,{}_2F_1\left(\begin{matrix}-k,n+s-k\\ 1+\frac{n}{2}+s-k+r\end{matrix}\middle|\frac{1}{2}\right),$$

and then use the Pfaff transform [BE, §2.1 (22)] in the form

$${}_2F_1\left(\begin{matrix}-k,n+s-k\\ 1+\frac{n}{2}+s-k+r\end{matrix}\middle|\frac{1}{2}\right)=2^{-k}\,{}_2F_1\left(\begin{matrix}-k,1-\frac{n}{2}+r\\ 1+\frac{n}{2}+s-k+r\end{matrix}\middle|-1\right).$$

Inserting this into (19) we get the right-hand side of (18). □

Remark 12. It must be pointed out that the series in (18) converges for $2n+s > 2k+1$, but it is guaranteed to be positive only for $s > -1$. For $s \le -1$ we cannot (in general) be sure that the right hand side of (18) is nonzero, which is essential for the asymptotic equality to hold.

Here are some more details. Denote the right hand side of (18) by $L_k(s)$, so that for $s > -1$ we have by Corollary 11

$$I_{m,k} \sim L_k(s) \qquad \text{as } m \to \infty.$$

The function $L_k(s)$ can also be expressed as follows:

$$L_k(s) = \frac{\Gamma(\frac{n}{2})\Gamma(1+\frac{n}{2}+s)}{m^{s+1-2k}} \frac{\Gamma(n+s-k)^2}{\Gamma(n-1)^2} \sum_{j=0}^{\infty} \frac{(1-\frac{n}{2})_j^2 (s+1)_j}{j!} {}_2\mathbf{F}_1\left(\begin{matrix} -k, 1-\frac{n}{2}+j \\ 1+\frac{n}{2}+s-k+j \end{matrix} \middle| -1 \right)^2,$$

where

$${}_2\mathbf{F}_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) := \frac{1}{\Gamma(c)} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{\Gamma(c+k)k!} x^k$$

is the regularized hypergeometric function, which is an entire function in all three parameters a, b, c. Also, as discussed before, the series converges for $2n+s > 2k+1$. Therefore

$$L_k(s) = m^{2k-s-1}\Gamma(1+\frac{n}{2}+s)\Gamma(n+s-k)^2 \times (\text{a holomorphic function in } \operatorname{Re} s > 2k+1-2n).$$

The function $L_k(s)$ is therefore holomorphic whenever $\operatorname{Re} s > 2k+1-2n$ and s avoids the singularities of $\Gamma(1+\frac{n}{2}+s)\Gamma(n+s-k)$. We have that $L_k(s) > 0$ for $s > -1$ but, indeed, for other values of s the function $L_k(s)$ need not be positive in general. In fact, it is easy to see that

$$L_0(-2) = 0,$$

and hence for $s = -2$ it is not true that $I_{m,0} \sim L_0(-2)$ as $m \to \infty$. □

Remark 13. For $k = 0$ we get the simpler formula

$$L_0(s) = \frac{\Gamma(\frac{n}{2})\Gamma(1+\frac{n}{2}+s)}{m^{s+1}} \frac{\Gamma(n+s)^2}{\Gamma(n-1)^2} {}_3\mathbf{F}_2\left(\begin{matrix} 1-\frac{n}{2}, 1-\frac{n}{2}, s+1 \\ 1+\frac{n}{2}+s, 1+\frac{n}{2}+s \end{matrix} \middle| 1 \right),$$

valid for $\operatorname{Re} s > 1-2n$. This agrees with the leading factor of I_m computed in [Ur] for $s > -1$. □

Proof of Proposition 4. The proof of Proposition 4 now follows upon combining Corollary 9 with the asymptotic expansions of $c_{j,k}(m)$, $c_{l,k}(m)$ (which are left to the reader, cf. Remark 6). Corollary 11 shows that, indeed, $A_0 > 0$. □

4. The H-harmonic Dirichlet space

As in [EY2], introduce the notation

$$I_m^\circ := \lim_{s \to -n} (s+n) I_m(s) = \begin{cases} (n-1)_m / \Gamma(m) & n \text{ even}, \\ 2(n-1)_m / \Gamma(m) & n \text{ odd}, \end{cases} \tag{20}$$

and consider the space

$$\mathcal{H}_\circ := \{ f = \sum_m f_m : \ f_m \in \mathbf{H}^m \forall m, \ \|f\|_\circ^2 := \sum_m I_m^\circ \|f_m\|_{\partial B^n}^2 < +\infty \}.$$

The quantity $\|f\|_\circ$ vanishes on constants, is a norm on $\mathcal{H}_{\circ,0} := \{ f \in \mathcal{H}_\circ : f(0) = 0 \}$, and a seminorm on $\mathcal{H}_\circ$; in other words, $\|f\|_\circ + |f(0)|$ is a norm on $\mathcal{H}_\circ$.

Let X_{jk}, $j,k = 1,\dots,n$, $j \neq k$, denote the tangential vector fields

$$X_{jk} := x_j\partial_k - x_k\partial_j$$

on $\mathbf{R}^n$, and denote by $\mathcal{X}_j$, $j = 1,\dots,n(n-1)$, the collection of all these operators (in some fixed order), i.e.

$$\{X_{jk} :\ j,k = 1,\dots,n,\ j \neq k\} = \{\mathcal{X}_j :\ j = 1,\dots,n(n-1)\}.$$

By a routine computation, one checks that

$$\sum_{j,k=1}^{n} X_{jk}^2 = 2\Delta_{\text{sph}},$$

where Δ_{sph} is the spherical Laplacian on $\mathbf{R}^n$: for $x = r\zeta$ with $r > 0$ and $\zeta \in \partial B^n$, the ordinary Euclidean Laplacian is expressed as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\text{sph}}.$$

The operator Δ_{sph} commutes with the action of the orthogonal group $O(n)$ of $\mathbf{R}^n$, hence it is automatically diagonalized by the Peter-Weyl decomposition (2): a simple computation reveals that

$$\Delta_{\text{sph}}|\mathbf{H}^m = -m(m+n-2)I|\mathbf{H}^m \tag{21}$$

where I stands for the identity operator.

In addition to the tangential operators X_{jk}, we denote by

$$\mathcal{R} := \sum_j x_j\partial_j$$

the radial derivative operator on $\mathbf{R}^n$; note that in the polar coordinates $x = r\zeta$, $\mathcal{R} = r\partial_r$. Finally, we denote by $\mathcal{Y}_j$, $0 \leq j \leq n(n-1)$, the collection of operators $\mathcal{Y}_j := \mathcal{X}_j$ for $j \geq 1$ and $\mathcal{Y}_0 := \mathcal{R}$:

$$\{\mathcal{R}\} \cup \{X_{jk} :\ j,k = 1,\dots,n,\ j \neq k\} = \{\mathcal{Y}_j :\ j = 0,\dots,n(n-1)\}, \qquad \mathcal{Y}_0 = \mathcal{R}.$$

Theorem 14. *If $f = \sum_m f_m$, $f_m \in \mathbf{H}^m$, is H-harmonic on B^n, $n > 2$, then the following assertions are equivalent:*

(a) $f \in \mathcal{H}_\circ$;
(b) $\sum_m m^{n-1}\|f_m\|^2_{\partial B^n} < +\infty$;
(c) *for some (equivalently, any) real* $p > \frac{n-1}{2}$, $(-\Delta_{\text{sph}})^{p/2} f \in L^2(B^n, d\rho_{2p-n})$, *i.e.*

$$\|(-\Delta_{\text{sph}})^{p/2} f\|^2_{2p-n} < +\infty; \tag{22}$$

(d) *for some (equivalently, any) integer* $p > \frac{n-1}{2}$,

$$\sum_{j_1,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_p} f\|^2_{2p-n} < +\infty. \tag{23}$$

If $n > 3$, then all the above are additionally equivalent to

(e) *for some (equivalently, any) integer p satisfying* $n-2 \geq p > \frac{n-1}{2}$,

$$\sum_{q=0}^{p}\sum_{j_1,\dots,j_q=0}^{n(n-1)} \|\mathcal{Y}_{j_1}\dots\mathcal{Y}_{j_q} f\|^2_{2p-n} < +\infty; \tag{24}$$

(f) *for some (equivalently, any) integer* p *satisfying* $n-2 \geq p > \frac{n-1}{2}$,

$$\sum_{|\alpha|\leq p} \|\partial^\alpha f\|^2_{2p-n} < +\infty. \tag{25}$$

If n *is odd, then* (a)–(d) *are additionally equivalent to*

(g) *for* $p = \frac{n-1}{2}$,

$$\sup_{0<r<1} \sum_{j_1,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_p} f(r\cdot)\|^2_{\partial B^n} < +\infty. \tag{26}$$

Furthermore, the square roots of the quantities in (b), (c), (d) *and* (g) *are equivalent to* $\|f\|_\circ$, *and of those in* (e), (f) *are equivalent to* $\|f\|_\circ + |f(0)|$.

Proof. (a) $\iff$ (b) Clearly from (20)

$$I_m^\circ \asymp m^{n-1}, \tag{27}$$

and the claim is thus immediate from the definition of $\|\cdot\|_\circ$.

(b) $\iff$ (d) Since the adjoint of X_{jk} in $L^2(\partial B^n, d\sigma)$ is just $-X_{jk}$, we have for any $g \in L^2(\partial B^n, d\sigma)$

$$\sum_{j=1}^{n(n-1)} \|\mathcal{X}_j g\|^2_{\partial B^n} = -\sum_{j,k=1}^{n} \langle X_{jk}^2 g, g\rangle_{\partial B^n} = -2\langle \Delta_{\text{sph}} g, g\rangle_{\partial B^n},$$

so for $g = \sum_m g_m$, $g_m \in \mathcal{H}^m$, as in (2),

$$\sum_{j=1}^{n(n-1)} \|\mathcal{X}_j g\|^2_{\partial B^n} = \sum_m 2m(m+n-2)\|g_m\|^2_{\partial B^n}, \tag{28}$$

by (21). Iterating this procedure, we get

$$\sum_{j_1,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_p} g\|^2_{\partial B^n} = \sum_m [2m(m+n-2)]^p \|g_m\|^2_{\partial B^n}.$$

Applying this now to $g(\zeta) = f(r\zeta)$ where f is H-harmonic on B^n, we obtain by (6)

$$\sum_{j_1,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_p} f(r\cdot)\|^2_{\partial B^n} = \sum_m [2m(m+n-2)]^p r^{2m} S_m(r^2)^2 \|f_m\|^2_{\partial B^n}, \tag{29}$$

and, for any $s > -1$,

$$\begin{aligned}
&\sum_{j_1,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_p} f\|^2_s \\
&\quad = \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)} \int_0^1 \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \sum_{j_1,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_p} f(r\cdot)\|^2_{\partial B^n} (1-r^2)^s r^{n-1}\,dr \\
&\quad = \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \int_0^1 \sum_m [2m(m+n-2)]^p \|f_m\|^2_{\partial B^n} t^{m+\frac{n}{2}-1} S_m(t)^2 (1-t)^s\,dt \\
&\quad = \sum_m [2m(m+n-2)]^p I_m(s) \|f_m\|^2_{\partial B^n} \qquad \text{by (4).}
\end{aligned}$$

Recalling (10), we see that if $s = 2p - n$, then for all $m \geq 1$

$$[2m(m+n-2)]^p I_m(s) \asymp m^{2p-s-1} = m^{n-1}, \tag{30}$$

while for $m = 0$ both sides vanish. This proves the claim.

(b) $\iff$ (c) First of all, from the expression of $d\rho_s$ in polar coordinates

$$d\rho_s(x) = \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} r^{n-1}(1-r^2)^s \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\, d\sigma(\zeta)\, dr, \qquad x = r\zeta,\ r > 0, \zeta \in \partial B^n,$$

we get the expression as Hilbert space tensor product

$$L^2(B^n, d\rho_s) = L^2((0,1), \tfrac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} r^{n-1}(1-r^2)^s\, dr) \otimes L^2(\partial B^n, \tfrac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\, d\sigma(\zeta)).$$

By (2) and (21), it follows that $\Delta_{\rm sph}$ then acts as

$$I \otimes \bigoplus_m [-m(m+n-2)] I | \mathcal{H}^m,$$

and, hence, $-\Delta_{\rm sph}$ is indeed a positive selfadjoint operator on each $L^2(B^n, d\rho_s)$, $s > -1$, and the power $(-\Delta_{\rm sph})^q$ makes sense for any real $q \geq 0$ by the spectral theorem:

$$(-\Delta_{\rm sph})^q \sum_m F_m(r) f_m(\zeta) = \sum_m [m(m+n-2)]^q F_m(r) f_m(\zeta)$$

for any $f_m \in \mathcal{H}^m$ and $F_m \in L^2((0,1), r^{n-1}(1-r^2)^s\, dr)$.

Using again (6), the claim thus follows by the same calculation as from (29) above.

(b) $\iff$ (g) Observe from (29), combined with the fact that $r^m S_m(r^2) \nearrow 1$ as $r \nearrow 1$, that the supremum on the left-hand side of (26) equals

$$\sum_m [2m(m+n-2)]^p \|f_m\|^2_{\partial B^n}$$

by the Lebesgue Monotone Convergence Theorem. Since now $2p = n-1$, the claim thus follows in the same way as in (30).

(b) $\iff$ (e) Observe first of all that the additional hypothesis $n > 3$ ensures that there exists at least one integer p in the interval $n - 2 \geq p > \frac{n-1}{2}$. Let

$$f(r\zeta) = \sum_m r^m S_m(r^2) f_m(\zeta), \qquad 0 \leq r < 1, \zeta \in \partial B^n,$$

be the Peter-Weyl decomposition of our H-harmonic function f. Then

$$\mathcal{X}_{j_1} \ldots \mathcal{X}_{j_l} \mathcal{R}^k f(r\zeta) = \sum_m \mathcal{R}^k [r^m S_m(r^2)] \mathcal{X}_{j_1} \ldots \mathcal{X}_{j_l} f_m(\zeta)$$

(which also remains in force for any permutation of the order of the $\mathcal{X}_j$ and $\mathcal{R}$ on the left-hand side). Hence, similarly as for (29),

$$\sum_{j_1,\ldots,j_q=1}^{n(n-1)} \|\mathcal{X}_{j_1} \ldots \mathcal{X}_{j_q} \mathcal{R}^k f(r\cdot)\|^2_{\partial B^n} = \sum_m [2m(m+n-2)]^q (\mathcal{R}^k [r^m S_m(r^2)])^2 \|f_m\|^2_{\partial B^n},$$

and, for any $s > -1$,

$$\sum_{j_1,\ldots,j_q=1}^{n(n-1)} \|\mathcal{X}_{j_1} \ldots \mathcal{X}_{j_q} \mathcal{R}^k f\|^2_s$$

$$= \frac{2\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \sum_m [2m(m+n-2)]^q \|f_m\|^2_{\partial B^n} \int_0^1 (\mathcal{R}^k[r^m S_m(r^2)])^2 r^{n-1}(1-r^2)^s \, dr$$
$$= \frac{\Gamma(\frac{n}{2}+s+1)}{\Gamma(\frac{n}{2})\Gamma(s+1)} \sum_m [2m(m+n-2)]^q \|f_m\|^2_{\partial B^n} \int_0^1 ((2t\partial_t)^k[t^{m/2} S_m(t)])^2 t^{\frac{n}{2}-1}(1-t)^s \, dt$$
$$= \sum_m [2m(m+n-2)]^q I_{m,k}(s) \|f_m\|^2_{\partial B^n}$$

by (11). Using (12), we again see that if $q+k \le p < n-1$ and $s = 2p-n$, then for all $m \ge 1$

$$[2m(m+n-2)]^q I_{m,k}(s) \asymp m^{2q-s-1+2k} \le m^{2p-s-1} = m^{n-1} \asymp I^\circ_m; \tag{31}$$

and if $q+k = p < n-1$ and $s = 2p-n$, then for all $m \ge 1$ even

$$[2m(m+n-2)]^q I_{m,k}(s) \asymp m^{2q-s-1+2k} = m^{2p-s-1} = m^{n-1} \asymp I^\circ_m. \tag{32}$$

For $m = 0$, (32) still holds, and so does (31) if $0 < q+k$; for $m = q = k = 0$, the right-hand side of (31) becomes 1 while the left-hand side is zero. Consequently, for $q < p < n-1$,

$$\sum_{j_1,\dots,j_q=0}^{n(n-1)} \|\mathcal{Y}_{j_1} \dots \mathcal{Y}_{j_q} f\|^2_{2p-n} \lesssim \|f\|^2_\circ + \delta_{q0}|f(0)|^2,$$

while for $q = p < n-1$ even

$$\sum_{j_1,\dots,j_q=0}^{n(n-1)} \|\mathcal{Y}_{j_1} \dots \mathcal{Y}_{j_q} f\|^2_{2p-n} \asymp \|f\|^2_\circ.$$

The claim follows.

(f) $\implies$ (e) By the Leibnitz rule, $\mathcal{X}_{j_1} \dots \mathcal{X}_{j_p} f(x) = \sum_{|\alpha| \le p} P_\alpha(x) \partial^\alpha f(x)$, with some coefficient functions P_α that are bounded on B^n (in fact — they are polynomials). The claim is thus immediate from the triangle inequality.

(e) $\implies$ (f) Observe that at any $x \in B^n$, $x \ne 0$, the tangential vector fields $\mathcal{X}_j$ span (very redundantly) the entire tangent space to the sphere $|x|\partial B^n$; thus together with $\mathcal{R}$, they span the whole tangent space at x. It follows that, for any $q \in \mathbf{N}$, the derivatives $\partial^\alpha f$, $|\alpha| = q$, can be expressed as linear combinations of the derivatives $\mathcal{Y}_{j_1} \dots \mathcal{Y}_{j_q} f$, $1 \le j_1, \dots, j_q \le n(n-1)$; furthermore, the coefficients of these linear combinations can be chosen to be bounded away from the origin $x = 0$ (where all the $\mathcal{X}_j$ as well as $\mathcal{R}$ vanish). In other words, if we momentarily denote by χ the characteristic function of the annular region $\frac{1}{4} < |x| < 1$, then for any $s > -1$ and $q \in \mathbf{N}$,

$$\sum_{|\alpha|=q} \|\chi \partial^\alpha f\|^2_s \lesssim \sum_{j_1,\dots,j_q=1}^{n(n-1)} \|\mathcal{Y}_{j_1} \dots \mathcal{Y}_{j_q} f\|^2_s,$$

and, hence, for any $p \in \mathbf{N}$,

$$\sum_{q=0}^{p} \sum_{|\alpha|=q} \|\chi \partial^\alpha f\|^2_s \lesssim \sum_{q=0}^{p} \sum_{j_1,\dots,j_q=1}^{n(n-1)} \|\mathcal{Y}_{j_1} \dots \mathcal{Y}_{j_q} f\|^2_s. \tag{33}$$

To treat the remaining region $|x| < 1/4$, we use a "subharmonicity" argument. Recall that H-harmonic functions possess the mean-value property [St, Corollary 4.1.3]

$$f(a) = \int_{\partial B^n} f(\phi_a(r\zeta))\, d\sigma(\zeta)$$

for any $r \in (0,1)$, $a \in B^n$ and f H-harmonic on B^n. Integrating over $0 < r < \frac{1}{4}$ yields

$$f(a) = c_n \int_{|x|<1/4} f(\phi_a(x))\, dx,$$

where $c_n := 1/\int_{|x|<1/4} dx$. Changing the variable x to $\phi_a(x)$ gives

$$f(a) = \int_{|\phi_a(x)|<1/4} f(x) F(a,x)\, d\rho_s(x),$$

with

$$F(a,x) := c_n \frac{\pi^{n/2}\Gamma(s+1)}{\Gamma(\frac{n}{2}+s+1)} \operatorname{Jac}_{\phi_a}(x)(1-|x|^2)^{-s}$$

smooth on $B^n \times B^n$. Hence for any multiindex α,

$$\partial^\alpha f(a) = \int_{|\phi_a(x)|<1/4} f(x)\, \partial_a^\alpha F(a,x)\, d\rho_s(x).$$

If $|a| < \frac{1}{4}$, one easily checks from (35) below that $|\phi_a(x)| < \frac{1}{4}$ implies

$$1 - |x|^2 = 1 - |\phi_a(\phi_a(x))|^2 = \frac{(1-|a|^2)(1-|\phi_a(x)|^2)}{1 - 2\langle a, \phi_a(x)\rangle + |a|^2|\phi_a(x)|^2} \geq \frac{(1-4^{-2})^2}{(1+4^{-2})^2} = \frac{15^2}{17^2},$$

or $|x| < 8/17$. Since $\partial_a^\alpha F(a,x)$ is, thanks to the smoothness of F on $B^n \times B^n$, bounded on $|a| < \frac{1}{4}$ and $|x| < \frac{8}{17}$, we thus get

$$|\partial^\alpha f(a)| \leq C_\alpha \int_{|\phi_a(x)|<1/4} |f(x)|\, d\rho_s(x) \leq C_\alpha \|f\|_s$$

with some finite C_α independent of $|a| < 1/4$ and f. Hence

$$\|(1-\chi)\partial^\alpha f\|_s^2 \leq C_\alpha^2 \|1-\chi\|_s^2 \|f\|_s^2,$$

and, consequently,

$$\sum_{q=0}^{p} \sum_{|\alpha|=q} \|(1-\chi)\partial^\alpha f\|_s^2 \lesssim \|f\|_s^2.$$

Combining this with (33) and setting $s = 2p - n$, the claim follows.

This completes the proof of the theorem. □

Part (b) of the last theorem can be reformulated as follows. Consider the weak-maximal operator X acting from $L^2(\partial B^n)$ into the Cartesian product of $[n(n-1)]^p$ copies of $L^2(\partial B^n)$ by

$$g \longmapsto \{\mathcal{X}_{j_1} \dots \mathcal{X}_{j_p} g\}_{j_1,\dots,j_p=1}^{n(n-1)};$$

that is, the domain of X consists of all $g \in L^2(\partial B^n)$ for which all the $\mathcal{X}_{j_1} \dots \mathcal{X}_{j_p} g$ exist in the sense of distributions and belong to $L^2(\partial B^n)$. (In other words, $X = Y^*$ where Y is the restriction of the formal adjoint $X^\dagger$ of X to $\bigoplus^{[n(n-1)]^p} C^\infty(\partial B^n)$.) Then $f \in \mathcal{H}$ belongs to $\mathcal{H}_\circ$ if and only if $f(r\cdot) \to f^*$ as $r \nearrow 1$ in $L^2(\partial B^n)$ for some "boundary value" f^* of f, and $f^* \in \operatorname{Dom}(X)$. Furthermore, $\|Xf^*\|$ is a seminorm equivalent to $\|f\|_\circ$. (The reader is referred to Grellier and Jaming [GJ, Theorem A] for much more detailed discussion of the matters above and boundary values of H-harmonic functions in general.)

We use this reformulation in the proof of the next proposition for $n = 3$, writing for ease of notation just f instead of f^*.

5. Moebius invariance

Proposition 15. *The space $\mathcal{H}_\circ$ is Moebius invariant: $f \in \mathcal{H}_\circ$ implies $f \circ U \in \mathcal{H}_\circ$, $f \circ \phi_a \in \mathcal{H}_\circ$ for any $U \in O(n)$ and $a \in B^n$. Also, the composition operators $f \mapsto f \circ U$, $f \mapsto f \circ \phi_a$ are continuous on $\mathcal{H}_\circ$.*

Proof. For $O(n)$-invariance, both assertions are immediate from (2) and the definition of inner product in $\mathcal{H}_\circ$ — in fact, the composition operator $f \mapsto f \circ U$, $U \in O(n)$, is unitary.

So consider the composition with ϕ_a. Assume first that $n > 3$. By part (f) of the last theorem, it is then enough to show that

$$\sum_{|\alpha|\le p} \|\partial^\alpha (f\circ\phi_a)\|^2_{2p-n} \lesssim \sum_{|\beta|\le p} \|\partial^\beta f\|^2_{2p-n} \tag{34}$$

for some integer p satisfying $n-2 \ge p > \frac{n-1}{2}$ (thanks to the assumption that $n > 3$, such an integer exists). However, by the chain rule, we have

$$\partial^\alpha (f\circ\phi_a)(x) = \sum_{|\beta|\le|\alpha|} c_{\alpha\beta}(x)(\partial^\beta f)(\phi_a(x)),$$

where the coefficient functions $c_{\alpha\beta}$ are polynomials in the derivatives of ϕ_a; in particular, they are bounded on B^n. Furthermore, the Jacobian $\mathrm{Jac}_{\phi_a}(x)$ is likewise bounded on B^n for any fixed $a \in B^n$. Consequently, for any $s > -1$,

$$\|\partial^\alpha (f\circ\phi_a)\|^2_s \lesssim \sum_{|\beta|\le|\alpha|} \|\partial^\beta f\|^2_s.$$

Summing over all $|\alpha| \le p$ and setting $s = 2p - n$, (34) follows.

It remains to deal with the case $n = 3$. Here we can use part (g) of the last theorem: that is, we need to show that

$$\sum_{j_1,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\dots\mathcal{X}_{j_p}(f\circ\phi_a)\|^2_{\partial B^n}$$

is bounded by a constant multiple of the same sum for f in the place of $f \circ \phi_a$. Observe that the tangential vector-fields $\mathcal{X}_j$, $j = 1,\dots,n(n-1)$, span (very redundantly) the entire tangent space to ∂B^n. Thus for any differentiable function g on ∂B^n and any $x \in \partial B^n$, $\sum_j \|\mathcal{X}_j g(x)\|^2 \asymp \|\nabla_t\, g(x)\|^2$, the norm-square of the restriction $\nabla_t\, g(x)$ of the tensor $\nabla g(x)$ at x to the tangent space of ∂B^n in the sense of Riemannian geometry. Now for any vector field X on ∂B^n, one has $X(g\circ\phi_a) = d\phi_a(X)g$. Since ϕ_a maps the sphere ∂B^n onto itself, the derived map $d\phi_a$ maps the tangent space of ∂B^n into itself. Finally, $d\phi_a|\partial B^n$ is a smoothly varying map on the compact manifold ∂B^n (hence, in particular, so is its Jacobian). Consequently,

$$\|\nabla_t\,(g\circ\phi_a)\|^2_{\partial B^n} = \|d\phi_a(\nabla_t\,)g\|^2_{\partial B^n} \le C_a\|\nabla_t\, g\|^2_{\partial B^n}$$

with some finite C_a (independent of g). Iterating this argument, it transpires that

$$\|\nabla_t{}^p(g\circ\phi)\|^2_{\partial B^n} \le C_a^p\|\nabla_t\, g\|^2_{\partial B^n}.$$

Passing from ∇_t back to the $\mathcal{X}_j$, the last inequality reads

$$\sum_{j_1,j_2,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\mathcal{X}_{j_2}\dots\mathcal{X}_{j_p}(g\circ\phi_a)\|^2_{\partial B^n} \le C_a^p \sum_{j_1,j_2,\dots,j_p=1}^{n(n-1)} \|\mathcal{X}_{j_1}\mathcal{X}_{j_2}\dots\mathcal{X}_{j_p}g\|^2_{\partial B^n},$$

which is what we needed to prove. □

Introduce the space

$$\mathcal{H}' := \{f \ H\text{-harmonic on } B^n : (s+n)\|f\|_s^2 \text{ has an analytic continuation in } s \text{ to a neighborhood of } s=-n\}.$$

By Proposition 1, $\mathcal{H}'$ contains the algebraic span of $\mathbf{H}^m$, $m \in \mathbf{N}$, and the square root $\|f\|'$ of the value of the analytic continuation above at $s=-n$ is a (semi-) norm on $\mathcal{H}'$, with the corresponding (semi-)inner product

$$\langle f,g\rangle' = \text{the analytic continuation to } s=-n \text{ of } (s+n)\langle f,g\rangle_s$$

coinciding with $\langle f,g\rangle_\circ$ on this span. It follows that $\mathcal{H}^\circ$ is just the completion of $\mathcal{H}'$ with respect to this inner product.

The following result shows that it may be appropriate to view $\mathcal{H}_\circ$ as the "H-harmonic Dirichlet space", and gives an answer to a question on p. 180 in [St].

Theorem 16. *The inner product in $\mathcal{H}_\circ$ is Moebius invariant:*

$$\langle f\circ\phi, g\circ\phi\rangle_\circ = \langle f,g\rangle_\circ$$

for any $f,g \in \mathcal{H}_\circ$ and ϕ in the Moebius group of B^n.

Proof. Since both $\mathcal{H}_\circ$ and its inner product $\langle\cdot,\cdot\rangle_\circ$ are $O(n)$-invariant (by their very construction), it is enough to prove the assertion for $\phi=\phi_a$; we can even assume that a is of the form $(a,0,\dots,0)\in B^n$ with some (abusing the notation) $0\le a<1$. Furthermore, since we know from the last proposition that the composition operator $f\mapsto f\circ\phi_a$ is continuous on $\mathcal{H}_\circ$, it is further enough to prove the assertion for f,g in a dense subset of $\mathcal{H}_\circ$. In particular, by linearity, we may assume that $f\in\mathbf{H}^m$ and $g\in\mathbf{H}^{m'}$ for some $m,m'\in\mathbf{N}$.

We will show that under all these hypotheses, $\langle f\circ\phi_a, g\circ\phi_a\rangle'$ exists for all $0\le a<1$ and does not depend on a. By the observations in the paragraph before the theorem, this will complete the proof.

Fix $0<\rho<1$. Recall that the measure

$$d\tau(x) := \frac{dx}{(1-|x|^2)^n}$$

on B^n is invariant under ϕ_a, and also

$$1-|\phi_a(x)|^2 = \frac{(1-|a|^2)(1-|x|^2)}{[x,a]^2}. \tag{35}$$

By the change of variable $x\mapsto\phi_a(x)$, we thus have, for any $s>-1$,

$$\begin{aligned}\langle f\circ\phi_a, g\circ\phi_a\rangle_s &= \frac{\Gamma(\frac n2+s+1)}{\pi^{n/2}\Gamma(s+1)}\int_{B^n}(f\overline{g})(\phi_a(x))(1-|x|^2)^{s+n}\,d\tau(x)\\ &= \frac{\Gamma(\frac n2+s+1)}{\pi^{n/2}\Gamma(s+1)}\int_{B^n}(f\overline{g})(x)(1-|\phi_a(x)|^2)^{s+n}\,d\tau(x)\end{aligned}$$

$$= \int_{B^n} (f\overline{g})(x) \Big(\frac{1-a^2}{1-2ax_1+|x|^2a^2} \Big)^{n+s} \, d\rho_s(x).$$

Passing to the polar coordinate $x = r\zeta$, with $0 \le r < 1$ and $\zeta \in \partial B^n$, we can continue with

(36)

$$= \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)} \int_0^1 \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_{\partial B^n} (f\overline{g})(r\zeta) \Big(\frac{1-a^2}{1-2ar\zeta_1+r^2a^2} \Big)^{n+s} (1-r^2)^s r^{n-1} \, d\zeta \, dr,$$

that is, using (6),

$$= \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)} \int_0^1 G(a,r)(1-r^2)^s r^{n-1} \, dr,$$

$$\text{where } G(a,r) := \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} r^{m+m'} S_m(r^2) S_{m'}(r^2) \int_{\partial B^n} (f\overline{g})(\zeta) \Big(\frac{1-a^2}{1-2ar\zeta_1+r^2a^2} \Big)^{n+s} \, d\zeta.$$

Carrying out the ζ integration shows that $G(a,r)$ is a holomorphic function of $|a| < \rho$ and $|r| < 1/\rho$.

Recall now that if $F(t) = \sum_{j=0}^\infty F_j(1-t)^j$ is holomorphic in some neighborhood of $t = 1$ and continuous on $[0,1]$, then

$$\mathcal{I}(s) := \int_0^1 F(t)(1-t)^s \, dt, \qquad s > -1, \tag{37}$$

extends to a holomorphic function of s on the entire complex plane $\mathbf{C}$, except for simple poles at $s = -j-1$, $j = 0,1,2,\dots$, with residues F_j. Differentiating (37) under integral sign, it transpires also that, for any $k = 0,1,2,\dots$,

$$\mathcal{I}_k(s) := \int_0^1 F(t) \Big(\log \frac{1}{1-t} \Big)^k (1-t)^s \, dt, \qquad s > -1,$$

extends to a holomorphic function of s on the entire complex plane $\mathbf{C}$, except for poles of multiplicity $k+1$ at $s = -j-1$, $j = 0,1,2,\dots$, with principal part $k!F_j/(s+j+1)^{k+1}$. See Lemma 2 in [EY2] for the details.

Now by [BE, §2.10 (12)], for $n > 2$ odd S_m is of the form $F_1(t) + F_2(t)(1-t)^{n-1} \log \frac{1}{1-t}$, with F_1, F_2 as in (37) (and S_m is actually a polynomial in $1-t$ for even n). It thus follows from the observation in the preceding paragraph that $\langle f \circ \phi_a, g \circ \phi_a \rangle_s$ extends to a holomorphic function of $|a| < \rho$ and $s \in \mathbf{C}$, except for at most simple poles at $s = -n, -n-1, \dots, -2n+2$ and at most double poles at $s = -2n-j+1$, $j \in \mathbf{N}$. Consequently, the function $(s+n)\langle f \circ \phi_a, g \circ \phi_a \rangle_s$ extends to a holomorphic function of $|a| < \rho$ and $s \in \mathbf{C}$ except for poles as above, excluding $s = -n$ where it assumes a finite value. In particular (taking $f = g$), this means that $f \circ \phi_a, g \circ \phi_a \in \mathcal{H}'$ for all $0 \le a < \rho$, and the inner product $\langle f \circ \phi_a, g \circ \phi_a \rangle'$ is a smooth function of these a.

Finally, it is legitimate to differentiate under the integral sign in (36), yielding, for $s > -1$,

(38)

$$\frac{\partial}{\partial a} \langle f \circ \phi_a, g \circ \phi_a \rangle_s = \frac{\Gamma(\frac{n}{2}+s+1)}{\pi^{n/2}\Gamma(s+1)} \int_0^1 \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_{\partial B^n} (f\overline{g})(\zeta) r^{m+m'} S_m(r^2) S_{m'}(r^2) \times$$

$$(n+s) \Big(\frac{1-a^2}{1-2ar\zeta_1+r^2a^2} \Big)^{n+s-1} \Big[\frac{\partial}{\partial a} \frac{1-a^2}{1-2ar\zeta_1+r^2a^2} \Big] (1-r^2)^s r^{n-1} \, d\zeta \, dr.$$

Repeating the argument above, it transpires that for all $0 \le a < \rho$,

$$\frac{\partial}{\partial a}(n+s)\langle f\circ\phi_a, g\circ\phi_a\rangle_s = (n+s)F_a(s), \tag{39}$$

where $F_a(s)$ is a holomorphic function of s except for at most double poles at $s = -2n+1-j$, $j \in \mathbf{N}$, and at most simple poles at $s = -n-1, \dots, -2n+2$; in particular, $F_a(s)$ is holomorphic near $s = -n$ and assumes a finite value there. Hence, thanks to the factor $n+s$ in (39),

$$\frac{\partial}{\partial a}\langle f\circ\phi_a, g\circ\phi_a\rangle' = 0 \quad \text{for } 0 \le a < \rho.$$

Since ρ was arbitrary, it follows that $\langle f\circ\phi_a, g\circ\phi_a\rangle' = \langle f\circ\phi_0, g\circ\phi_0\rangle' = \langle f, g\rangle'$ for all $0 \le a < 1$, completing the proof. □

The last proof is modeled on the proof of Theorem 15 in [EY2].

6. Reproducing kernel

The reproducing kernel of $\mathcal{H}_\circ$ — more precisely, for its subspace $\mathcal{H}_{\circ,0}$ of functions vanishing at the origin — is given by

$$K_\circ(x,y) = \sum_{m>0} \frac{S_m(|x|^2)S_m(|y|^2)}{I_m^\circ} Z_m(x,y). \tag{40}$$

In this section, we give a closed expression for this sum.

Proposition 17. *The kernel $K_\circ(x,y)$ obeys the following transformation formula:*

$$K_\circ(x,y) = K_\circ(\phi_a(x),\phi_a(y)) - K_\circ(\phi_a(x),a) - K_\circ(a,\phi_a(y)) + K_\circ(a,a). \tag{41}$$

Proof. Let $\{e_j\}$ be an orthonormal basis for $\mathcal{H}_{\circ,0}$; thus by the standard formula for reproducing kernel [Aro]

$$K_\circ(x,y) = \sum_j e_j(x)\overline{e_j(y)}. \tag{42}$$

By Theorem 16, $e_j\circ\phi_a$ will also be mutually orthogonal unit vectors in $\mathcal{H}_\circ$, for any $a \in B^n$; since constants are orthogonal to everything in $\mathcal{H}_\circ$, it follows that $\{e_j\circ\phi_a - e_j(a)\}$ is also an orthonormal basis for $\mathcal{H}_{\circ,0}$, whence again by [Aro]

$$K_\circ(x,y) = \sum_j (e_j\circ\phi_a(x) - e_j(a))\overline{(e_j\circ\phi_a(y) - e_j(a))}.$$

Expanding the right-hand side and using (42), we get (41). □

Introduce the function

$$f_n(x) := \int_0^x (1-t)^{n-2} {}_2F_1\left(\begin{matrix} n, \frac{n}{2} \\ 1+\frac{n}{2}\end{matrix}\Big| t\right) dt. \tag{43}$$

Proposition 18. $K_\circ(x,x) = c_n f_n(|x|^2)$ *for some constant c_n depending only on n.*

Proof. Writing for brevity $K_\circ(x,x) =: K(x)$, setting $y = x$ in (41) yields

$$K(x) = K(\phi_a(x)) - K_\circ(\phi_a(x),a) - K_\circ(a,\phi_a(x)) + K(a).$$

The last three terms are H-harmonic functions; applying Δ_h to both sides, we thus obtain

$$\Delta_h K = \Delta_h(K\circ\phi_a) = (\Delta_h K)\circ\phi_a$$

by the Moebius invariance of Δ_h. Since $a \in B^n$ is arbitrary, it follows that

$$\Delta_h K \equiv c_n \tag{44}$$

for some constant c_n depending only on n. Now K is evidently a radial function, $K(x) =: f(|x|^2)$ for some function f on $(0,1)$. Since, by a short computation,

$$\Delta_h f(|x|^2) = (1-|x|^2)[(1-|x|^2)(2nf'(|x|^2)+4|x|^2 f''(|x|^2))+2(n-2)2|x|^2 f'(|x|^2)],$$

we see that $f(t)$ must satisfy the differential equation

$$(1-t)[(1-t)(2nf'+4tf'')+4(n-2)tf'] \equiv c_n,$$

that is,

$$\frac{(1-t)^n}{t^{n/2-1}}\Big(\frac{t^{n/2}}{(1-t)^{n-2}}f'(t)\Big)' \equiv c_n, \tag{45}$$

with the initial condition $f(0)=0$ (since $K_\circ(0,0)=0$). This gives

$$f(t) = c_n \int_0^t (1-t)^{n-2} {}_2F_1\Big(\begin{matrix}\frac{n}{2}, n\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big)\, dt,$$

i.e. $f(t) = c_n f_n(t)$. □

Proposition 19. *We have*

$$f_n(x) = \frac{n-2}{2(n-1)} x\, {}_3F_2\Big(\begin{matrix}2-\frac{n}{2}, 1, 1\\ 2, 1+\frac{n}{2}\end{matrix}\Big| x\Big) + \frac{n}{2(n-1)} \log\frac{1}{1-x}. \tag{46}$$

Proof. Using the Euler transform we can express the integrand in (43) as

$$(1-t)^{n-2} {}_2F_1\Big(\begin{matrix}n, \frac{n}{2}\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big) = (1-t)^{-1} {}_2F_1\Big(\begin{matrix}1-\frac{n}{2}, 1\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big).$$

Next we are going to use the well known contiguous relation (see [Olv, 15.3.13])

$$(c-a-b) {}_2F_1\Big(\begin{matrix}a, b\\ c\end{matrix}\Big| t\Big) + a(1-t) {}_2F_1\Big(\begin{matrix}a+1, b\\ c\end{matrix}\Big| t\Big) - (c-b) {}_2F_1\Big(\begin{matrix}a, b-1\\ c\end{matrix}\Big| t\Big) = 0.$$

For $a = 1-n/2$, $b=1$, $c = 1+n/2$ this translates into

$$(n-1) {}_2F_1\Big(\begin{matrix}1-\frac{n}{2}, 1\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big) = -(1-\frac{n}{2})(1-t) {}_2F_1\Big(\begin{matrix}2-\frac{n}{2}, 1\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big) + \frac{n}{2} {}_2F_1\Big(\begin{matrix}1-\frac{n}{2}, 0\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big).$$

Consequently,

$$(1-t)^{-1} {}_2F_1\Big(\begin{matrix}1-\frac{n}{2}, 1\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big) = -\frac{2-n}{2(n-1)} {}_2F_1\Big(\begin{matrix}2-\frac{n}{2}, 1\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big) + \frac{n}{2(n-1)}\frac{1}{1-t}$$

and

$$f(x) = \int_0^x (1-t)^{-1} {}_2F_1\Big(\begin{matrix}1-\frac{n}{2}, 1\\ 1+\frac{n}{2}\end{matrix}\Big| t\Big)\, dt = \frac{(n-2)x}{2(n-1)} {}_3F_2\Big(\begin{matrix}1, 2-\frac{n}{2}, 1\\ 2, 1+\frac{n}{2}\end{matrix}\Big| x\Big) + \frac{n}{2(n-1)} \log\frac{1}{1-x},$$

proving (46). □

Theorem 20. *The reproducing kernel is given by*

$$K(x,y) = \frac{2(n-1)}{n}[f_n(|x|^2) + f_n(|y|^2) - f_n(|\phi_y(x)|^2)], \tag{47}$$

with f_n as in (46).

Proof. Replacing x, y in (41) by $\phi_a(x)$, $\phi_a(y)$, respectively, and setting $y = x$, we get (recall that $\phi_a \circ \phi_a = \mathrm{id}$)

$$K_\circ(\phi_a(x), \phi_a(x)) = K_\circ(x,x) - K_\circ(x,a) - K_\circ(a,x) + K_\circ(a,a),$$

that is, replacing a by y,

$$K_\circ(x,y) = \frac{K_\circ(x,x) + K_\circ(y,y) - K_\circ(\phi_y(x), \phi_y(x))}{2}$$

(recall that $K_\circ(x,y)$ is real-valued and $K_\circ(y,x) = K_\circ(x,y)$ — a reflection of the fact that H-harmonicity is preserved by complex conjugation). By the last two propositions, this gives

$$K_\circ(x,y) = \frac{c_n}{2}[f_n(|x|^2) + f_n(|y|^2) - f_n(|\phi_y(x)|^2)].$$

It remains to compute the constant c_n. To this end, we momentarily return to the function $K_\circ(x,x) = K(x) = c_n f_n(|x|^2)$ above, and compare the behavior at the origin of

$$K_\circ(x,x) = c_n f_n(|x|)^2 \tag{48}$$

and

$$K_\circ(x,x) = \sum_{m>0} \frac{S_m(|x|^2)^2}{I_m^\circ} Z_m(x,x) \tag{49}$$

(cf. (40)). Both (48) and (49) are functions of $t := |x|^2$, and the derivative of (48) with respect to t at $t = 0$ equals c_n (since $f_n'(0) = 1$). For the analogous derivative of (49) at $t = 0$, the only contribution comes from the term $m = 1$ (the terms with $m > 1$ have double or higher order zeros at $t = 0$), and is easily seen to be equal to $4(n-1)/n$. Thus $c_n = 4(n-1)/n$, completing the proof. □

It seems quite difficult to obtain (47) by directly summing (40).

7. Concluding remarks

In the holomorphic, pluriharmonic and harmonic cases, the analogues of the $I_m(s)$ are actually continuous and positive for all $s > -n-1$ ("the Wallach set"), hence one takes just the limit instead of the analytic continuation. The H-harmonic case seems to be fundamentally different: as s decreases, $I_m(s)$ become 1 when $s = -1$ (this is no surprise — the Hardy space situation), then seem again to become all 1 at $s = -2$ (this is already quite unexpected and definitely has no parallel in the above three cases), and a little thereafter become negative (!) in general, only to "re-surface" into our Dirichlet space at $s = -n$. Furthermore — this is again highly surprising and without any analogue in the three cases above — our Dirichlet space seems to appear once again at $s = -n-1$: namely, the residue of $I_m(s)$ at $s = -n-1$ is the same as at $s = -n$, for all $m \in \mathbf{N}$. Understanding the Wallach set, i.e. the set of all $s \in \mathbf{C}$ for which $1/I_m(s) \geq 0$ $\forall m$, would be highly desirable.

More concretely, by analogy with the situation for spaces of holomorphic functions on bounded symmetric domains in $\mathbf{C}^n$ (see e.g. [Ara]), let us define the *continuous H-harmonic Wallach set* by

$$\mathcal{W}_c := \{s \in \mathbf{C} : 0 < I_m(s) < +\infty \ \forall m \in \mathbf{N}\},$$

and the *discrete H-harmonic Wallach set of level k*, $k = 1, 2, \dots$, by

$$\mathcal{W}_d^k := \{s \in \mathbf{C} : I_m \text{ has a pole at } s, \text{ and the } -k\text{-th Laurent coefficients} \\ \text{are of the same sign, } \forall m \in \mathbf{N}\}.$$

(Thus $-n, -n-1 \in \mathcal{W}_d^1$.) For each $s \in \mathcal{W}_c$, we have the associated reproducing kernel space $\mathcal{H}_s$ (an "analytic continuation" of our weighted H-harmonic Bergman spaces); and for each $s \in \mathcal{W}_d^k$, one can construct an analogue of our Dirichlet space (see [EY2] for the M-harmonic situation with $k=2$).

Question 1. What is $\mathcal{W}_c$, and what are $\mathcal{W}_d^k$, $k = 1, 2, \dots$?

Question 2. Are there any more cases where $I_m(s_1) = I_m(s_2)$ $\forall m$ with $s_1 < s_2$, like the above situation $s_1 = -1$, $s_2 = -2$? Are there any more cases when I_m would have poles of the same order and strength for all m at two different values of s, like the above situation at $s = -n$ and $s = -n-1$?

Note also that (10), while valid for all $s \geq -1$, fails for $s = -2$ (since $I_m(-2) = 1$ for all m).

Question 3. What is the behavior of $I_m(s)$ as $m \to \infty$, for general fixed $s \in \mathbf{C}$?

Quite generally, for any real s we can introduce the spaces

$$\mathcal{H}_{\#s} := \{f = \sum_m f_m \ H\text{-harmonic on } B^n : \ \|f\|^2_{\#s} := \sum_m (m+1)^{-s-1} \|f_m\|^2_{\partial B^n} < +\infty\},$$

(where $f_m \in \mathbf{H}^m$ are, as before, the Peter-Weyl components of f). Clearly $\mathcal{H}_{\#s}$ are Banach spaces, and $\mathcal{H}_{\#s_1} \subset \mathcal{H}_{\#s_2}$ continuously for $s_1 < s_2$. Also by (10),

$$\mathcal{H}_{\#s} = \mathcal{H}_s \qquad \text{for } s > -1,$$

with equivalent norms; while by (27),

$$\mathcal{H}_{\#-n} = \mathcal{H}_\circ,$$

our H-harmonic Dirichlet space, with the norm $\|f\|_{\#-n}$ equivalent to $\|f\|_\circ + |f(0)|$. On the other hand, the observations in the preceding paragraphs show that $\mathcal{H}_{-2}$ does not coincide with $\mathcal{H}_{\#-2}$, but instead $\mathcal{H}_{-2} = \mathcal{H}_{\#-1}$ is again the H-harmonic Hardy space of Stoll [St2]. Understanding the spaces $\mathcal{H}_{\#s}$ for general $s \in \mathbf{R}$ is tantamount to having the answer to Question 3 above.

Our last remark concerns the semi-inner product

$$\langle f, g\rangle_\circ = \sum_m I_m^\circ \langle f, g\rangle_{\partial B^n} \tag{50}$$

in our Dirichlet space $\mathcal{H}_\circ$.

Question 4. Is (50) the unique (up to constant multiple) Moebius-invariant inner product on H-harmonic functions on B^n?

For the holomorphic case, the corresponding assertion is true [Zh, Theorem 6.16]; in the harmonic case it does not make sense, and for the M-harmonic case it is wide open.

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