The regularity theory for area minimizing currents in codimension higher than 1

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Plateau's problem

Problem

Find the surface of least area spanning a given "contour" (in a Euclidean space or more generally in a Riemannian manifold Σ). Originally for 2*d* surfaces in $\mathbb{R}^3$: here in general dimension and codimension.

Some notation:

- m will be the dimension of the surfaces ($m-1$ that of the "contour")
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What is a current?

Following De Rham an m-dimensional current T is a linear map on the space of smooth (compactly supported) m-forms ω:

$$\omega \quad \mapsto \quad T(\omega) \in \mathbb{R}\,.$$

- We recover classical C^1 oriented surfaces Γ via integration

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- We define boundaries "forcing" Stokes theorem:

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From classical linear functional analysis

Theorem

Given a smooth oriented closed $m-1$-dimensional surface $\Gamma \subset \mathbb{R}^{m+n}$ there is an m-dimensional current T which minimizes the mass $\mathbf{M}$ among those with $\partial T = \Sigma$.

Problem: our generalized solution might have real multiplicity. In fact there are much more severe problems: a foliation $\{\Sigma_t\}$ by smooth surfaces defines naturally a current:

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Integer rectifiable currents

Federer and Fleming '60: restrict the class of generalized surfaces. An integer rectifiable current consists of a (countable) collection of

- Γ_i C^1 oriented surfaces
- $K_i \subset \Gamma_i$ pairwise disjoint compact subsets
- k_i positive integers

with

$$\sum_i k_i \mathrm{Vol}^n(K_i) < \infty \qquad (1)$$

The action on forms is given by

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The FF theory in a nutshell

A "hard" compactness theorem (no linear structure anymore!):

Corollary

Given a smooth oriented closed $m-1$-dimensional surface $\Gamma \subset \mathbb{R}^{m+n}$ there is an m-dimensional i.r. current T which minimizes the mass among all those with $\partial T = \Sigma$.

A suitable approximation algorithm with classical piecewise smooth surfaces (the so-called "deformation lemma")

Corollary

If there is a minimizer in the class of piecewise smooth surfaces, this is a minimum among integer rectifiable currents.

Last but not least: the FF theory is homological, which makes it a very flexible tool to study geometric and sometimes also topological questions.

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Minimizers might be singular.

Theorem (De Giorgi + Federer + Fleming + Almgren + Simons, 60 to 70)

In codimension 1 a minimizer is a regular submanifold except for a closed set of dimension at most $m - 7$. And rectifiable: Simon.

Theorem (Almgren 80)

In higher codimension a minimizer is a regular submanifold except for a closed set of dimension at most $m - 2$.

The codimension 1 result has a large amount of applications to geometric problems and was the starting point of powerful generalizations (regularity theory for stable surfaces, etc.).
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Theorem (Chang 88)

The singular set of 2-dimensional area minimizing currents is discrete.

Chang's proof starts from assuming the existence of a "branched center manifold".

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Optimality

Both dimension bounds are optimal:

- The celebrated Simons' cone in $\mathbb{R}^8$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2 \, .$$

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Two bad guys

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A "new proof"

After a few years of effort, in five joint works with Emanuele Spadaro we have written a shorter (and, we hope, much more readable) proof of Almgren's theorem.

- Same program of Almgren (see four main steps below).
- Some core ideas and in particular the main hard estimate (frequency function, see in a while)
- Several new techniques of which we take advantage;
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Theorem (De Lellis -Spadaro - Spolaor, 2015)

Chang's result is valid for two suitable classes of almost minimizing currents (semicalibrated currents and spherical cross sections of 3-dimensional area minimizing cones).

Answer to a question of Rivière and Tian, particular cases covered previously by Rivière - Bellettini and Bellettini.

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Almgren's theorem is valid for semicalibrated currents.

(Hopefully) soon: boundary regularity (open question: last remark of Almgren's book!) .

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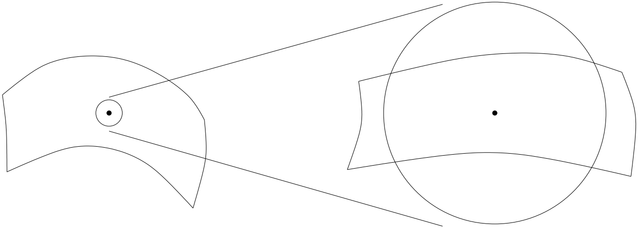
Step 0: tangent planes

Rescale homothetically a smooth surface keeping fixed a point x_0: the corresponding rescalings converge to a plane (the tangent to the surface at that point).



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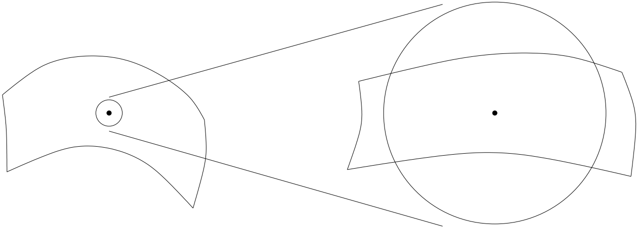

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Step 0: tangent planes I

For an area-minimizing current of dimension m there is at least one subsequence of rescalings converging to a plane (in a suitable weak sense) except when x_0 belongs to an exceptional set of dimension at most $m-2$.

It is a corollary of a powerful generalization (Almgren's stratification, due to Almgren!) of the so-called Federer reduction argument. Well absorbed in the literature and very short proof, used by a few authors in different contexts (see e.g. Simon, White, Wickramasekera).

Beware: the limiting plane might "pick" multiplicity. E.g. the bad guy

$$\Sigma_\varepsilon := \left\{(z,w) \in \mathbb{C}^2 : (\varepsilon z)^2 = (\varepsilon w)^3\right\}$$

converges to a double copy of the plane $\{z = 0\}$.

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Codimension 1: De Giorgi ε-regularity theory

Theorem (De Giorgi, codimension 1)

Once an area-minimizing current is sufficiently close to a plane in a ball, it must be a regular surface in half that ball.

Almgren, sixties: In general codimension De Giorgi's theorem is true provided the plane has multiplicity 1 (and the bad guy is a counterexample as soon as the multiplicity is 2).

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De Giorgi's idea

Consider a graph (in any codimension) $\{(x, f(x)) : x \in \Omega\}$. The volume of the graph is

$$\int_\Omega \sqrt{1 + |Df|^2 + \text{squares of minors}} = \int_\Omega \left(1 + \frac{|Df|^2}{2} + O(|Df|^4)\right) .$$

Thus a minimal graph is close to the graph of an harmonic function when $|Df| << 1$.

Harmonic functions have very strong decay of integral norms: if you can approximate efficiently an area minimizing current with an harmonic graph you can show that its distance to the best approximating flat plane decays at smaller scales.

This is usually called excess decay, where the "excess" is a suitable (integral) quantity measuring the flatness of the current.

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De Giorgi's idea fails in higher codimension and higher multiplicity

The bad guy

$$\{(z,w) \in \mathbb{C}^2 : z^2 = w^3\}$$

is rather flat in small neighborhoods of $(0,0)$ but cannot be approximated with the graph of a (single-valued!) function.

Worse new: choose ε extremely small and consider

$$\Gamma = \{(z,w) \in \mathbb{C}^2 : z^2 = \varepsilon w\}.$$

At scale 1 this is very close to two copies of the plane $\{z = 0\}$. But between the scales 1 and ε the surface Γ becomes less flat: the "decay" starts at scale ε.

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Almgren's Step 1

The starting point of his program: build a theory of functions taking multiple values (a fixed number, say, Q: the multiplicity of the best planar approximation) and minimizing an appropriate generalized Dirichlet energy.

Main achievements:

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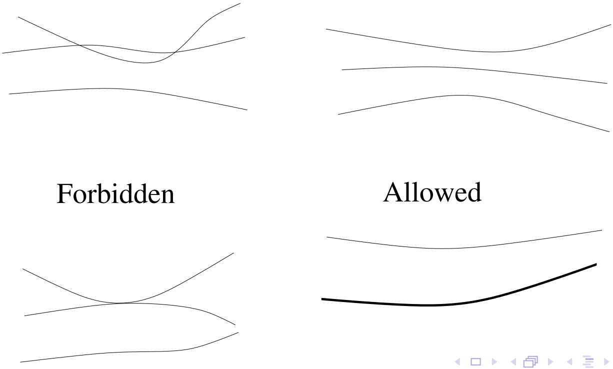
Every minimizer is Hölder in the interior and, except for a set of codimension 2 in the domain, in a suitable neighborhood of any other point it consists of Q harmonic sheets. Moreover any pair of these sheets are either disjoint or they coincide.



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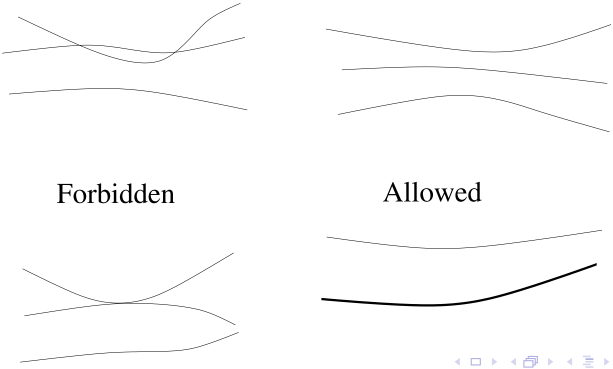

What's new?

We simplify and extend several steps of Almgren's theory.

We take advantage of new techniques in metric geometry and metric analysis to avoid some hard combinatorial arguments. This becomes very important in the later steps, where we merge a "metric" point of view for $W^{1,2}$ Q-valued maps with recent developments in the metric theory of currents (due to Jerrard-Soner, Ambrosio-Kirchheim and White).

We can prove new regularity results not stated in Almgren's book: for instance we prove higher integrability of the gradient (plays a very important role in Step 2, see below) and Hirsch extends the continuity up to the boundary.

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We take advantage of new techniques in metric geometry and metric analysis to avoid some hard combinatorial arguments. This becomes very important in the later steps, where we merge a "metric" point of view for $W^{1,2}$ Q-valued maps with recent developments in the metric theory of currents (due to Jerrard-Soner, Ambrosio-Kirchheim and White).

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What triggers the sheeting theorem?

Almgren's famous discovery: the monotonicity of the frequency function

$$I(r) := \frac{r \int_{B_r} |Du|^2}{\int_{\partial B_r} |u|^2}\,.$$

A very robust computation gives that $r \mapsto I(r)$ is increasing. For classical harmonic functions

$$I(0) = \lim_{r \downarrow 0} I(r)$$

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Two consequences of the mononicity of I:

- $I(0) < \infty$ is finite $\Rightarrow$ there is a first nontrivial expansion of u, a sort of "tangent function".
- The tangent function is homogeneous.

Two-dimensional homogeneous minimizers can be classified: if 0 is a singular point there is a "separation" of sheets in the punctured disk.

Thus for a planar 2-valued harmonic function all singular points (which must have multiplicity 2) are isolated.

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The frequency function is a limiting case of a family of "smooth" frequency functions, which are also monotone:

$$I_\varphi(r) = \frac{\int |Du(x)|^2 \varphi\left(\frac{|x|}{r}\right)\, dx}{\int -\frac{|u(x)|^2}{|x|}\varphi'\left(\frac{|x|}{r}\right)\, dx}\,.$$

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A first plan

We can conceive a first "blow-up" plan for the big theorem which starts as a contradiction argument, assuming the existence of an area minimizing integer rectifiable current with too many singular points.

To fix ideas let us assume that the multiplicity of a certain area-minimizing current T is (a.e.) either 1 or 2.

- By Step 0 we can try to prove regularity for most of the points where we have one "weak tangent plane";
- Fix such a point p and such a plane π: if π has multiplicity 1, then p is a regular point;
- Assume therefore the plane has multiplicity 2. If all points in a neighborhood of p has multiplicity 2, we can "divide" by 2 the current and reduce it to the case of multiplicity 1.
- We conclude that our current T must have many singular points p with a "weak tangent plane" of multiplicity 2, but always surrounded by at least some points of lower multiplicity (i.e. 1).

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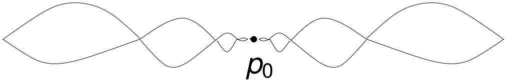
- A lot of these special points must accumulate around some p_0 where there is a weak tangent plane of multiplicity 2 (problem: the convergence to a plane of the rescalings might happen along a subsequence and the clustering of singularities along a different subsequence;).



A first plan III

- Part 1 of the plan is then to approximate the current at small scales with Lipschitz Q-valued maps which will almost minimize the Dirichlet energy.
- The second part of the plan is to rescale these approximations, prove their convergence to a (nontrivial!) harmonic limit and show that the latter “inherits” the large singular set of the current, contradicting the regularity theory for harmonic Q-valued maps.
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An approximation algorithm which combines the metric theory of currents of Ambrosio-Kirchheim (most notably a BV estimate by Jerrard and Soner) with the metric approach to $W^{1,2}$ Q-valued maps.

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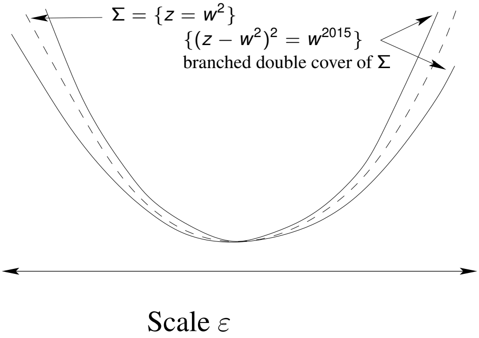
Unfortunately the plan fails miserably on the very bad guy:



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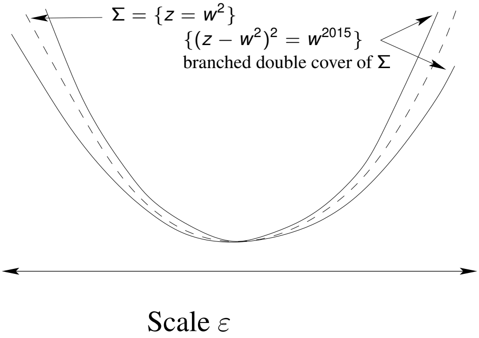

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It is not a minor technical point: in Almgren's version at this stage we would not even be at $1/4$ of the proof. The remaining 3/4 are needed to get around this point.

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Center manifold: Step 3

In the linear case, i.e. for harmonic Q-valued maps, we can simply subtract the regular "core" which is the average of the Q sheets (i.e. a a classical harmonic function).

This does not work in the nonlinear setting of area-minimizing currents. However, we can hope to at least approximate the average with sufficient accuracy with a smooth surface: this, following Almgren, will be called center manifold.

We cannot however "subtract" the center manifold (this will not give a harmonic approximation): rather we must approximate again the area-minimizing current with a (Q-valued) map taking values in the normal bundle of the center manifold.

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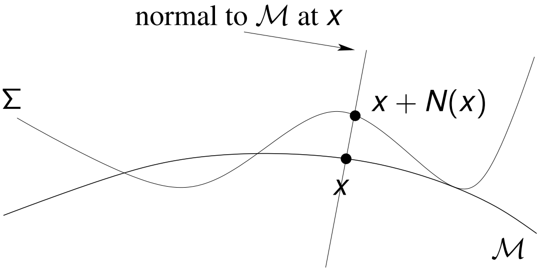
Taylor expansion again

$\mathcal{M} \subset \mathbb{R}^{m+n}$ (m-dimensional) smooth manifold.

$N : \mathcal{M} \to \mathbb{R}^{m+n}$ a (classical) map with the property that

$$x + N(x) \quad \perp \quad T_x\mathcal{M} \qquad \forall x$$

$\Sigma = \{x + N(x) : x \in \mathcal{M}\}$



Taylor expansion again II

Then,

$$\mathrm{Vol}^n(\mathcal{M}) = \mathrm{Vol}^n(\Sigma) + \int_{\mathcal{M}} \mathbb{H} \cdot N + \frac{1}{2}\int_{\mathcal{M}} |DN|^2 + \int_{\mathcal{M}} \mathbf{A}(N, N) + H.O.T.$$

where

- $\mathbb{H}$ is the mean curvature;
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Taylor expansion again III

The second fundamental form might be assumed very small: $\mathcal{M}$ is the "average" of the sheets of the current, which is assumed quite flat.

In our case N takes more than one value and the linear term must be substituted by

$$Q \int_{\mathcal{M}} \mathbb{H} \cdot \mathrm{Av}\,(N)$$

where $\mathrm{Av}\,(N)(x)$ is the average of the values taken by N at x: this is also small.

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Key properties of $\mathcal{M}$

Observe: $\mathcal{M}$ must be at least C^2 to carry on the computations above.

In fact we will need to take first variations in some arguments and this will require at least C^3 regularity.

The center manifold must be very close to the "average of the sheets". Assume for instance that in fact the current were a smooth manifold with multiplicity Q: in this case the center manifold must coincide with the current itself.

Corollary: whatever algorithm is used to produce $\mathcal{M}$, as a side effect it should give direct C^3 regularity in the "easy" assumption under which De Giorgi-Allard gives $C^{1,\alpha}$, without using Schauder theory

It is indeed possible to prove directly $C^{3,\alpha}$ with a "short elementary" proof (cf. a separate paper with Emanuele Spadaro).

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We derive these better estimates as a consequence of the higher integrability of the gradient of harmonic Q-valued maps u mentioned a few slides ago. Namely there is an estimate of type

$$\int_{B_{r/2}} |Du|^p \leq \left(\int_{B_r} |Du|^2 \right)^{2/p} \tag{2}$$

We can derive a suitable counterpart of (2) even in the nonlinear setting of area-minimizing currents. We still need to combine this latter estimate with a rather hard lemma of Almgren to derive the final approximation theorem, but (2) allows us to cut the most complicated part of Almgren's proof for Step 2.

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Note that (2) cannot be improved to an L^∞ bound: in fact it can be shown that p depends on both Q and m (for $Q \to \infty$, $p \to 2$).

The center manifold

Returning to Step 3, the center manifold is constructed with the following idea.

- Recall that we assume that the current is very close to a plane at a certain given scale, say 1, around a given point, say 0.
- Fix any point x and let r be the first scale around x at which the current is not anymore too close to a plane.
- At that scale we approximate the current with a (Lipschitz) Q-valued graph (over the best approximating plane!), using Step 2.
- Take the average of the sheets of this map and smooth it (for instance by convolution).
- Patch these local approximations together... somehow... into a single center manifold $\mathcal{M}$.
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- Patch these local approximations together... somehow... into a single center manifold $\mathcal{M}$.
- Last but not least "approximate" again the current with a Q-valued map on the normal bundle of $\mathcal{M}$.

The center manifold

Returning to Step 3, the center manifold is constructed with the following idea.

- Recall that we assume that the current is very close to a plane at a certain given scale, say 1, around a given point, say 0.
- Fix any point x and let r be the first scale around x at which the current is not anymore too close to a plane.
- At that scale we approximate the current with a (Lipschitz) Q-valued graph (over the best approximating plane!), using Step 2.
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What's new?

Hard to say, because Almgren's proof is over 500 pages long and even the statements are extremely intricate. This is however where our proof is much shorter (almost a factor 10).

The last approximation step is surely rather different since Almgren seem to start a new approximation procedure ex-novo. Our approach is instead to take locally the Lipschitz approximating maps of the "construction algorithm" and "reparameterize them" from the center manifold.

This requires a rather subtle change of coordinates theorem for Q-valued maps (the subtlety being in the estimate of certain integral quantities). However the theorem can be proved in a very effective way with the techniques we mentioned in Step 1 and from it the final approximation follows rather easily.

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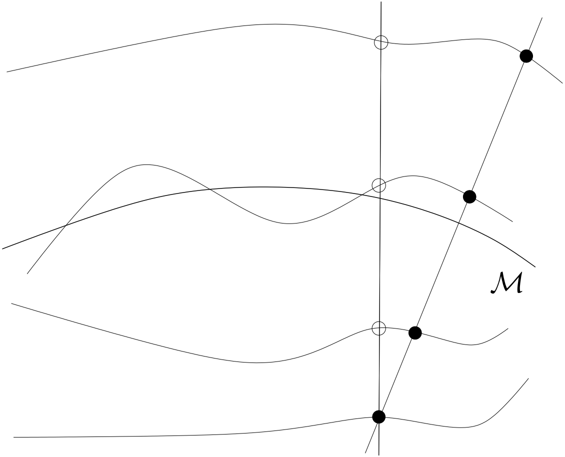
Changing coordinates is subtle!



What's new? II

In the construction algorithm we take advantage of the "splitting before tilting phenomenon". The terminology is borrowed from an important paper of Rivière, where he notices the following:

An area-minimizing 2-dimensional current which is close at a certain scale to a (multiple of a) plane can become less flat at a smaller scale only if the sheets "separate".

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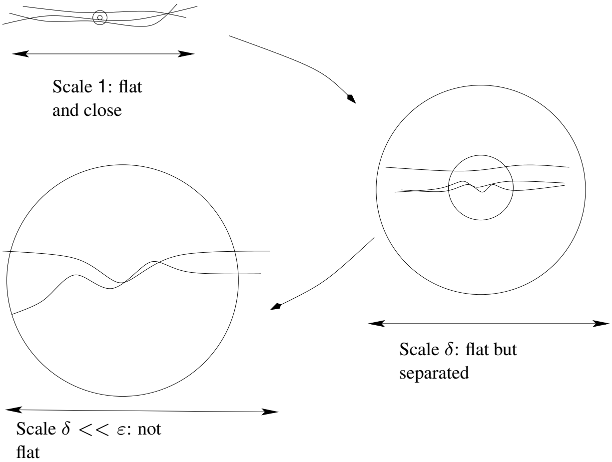

What’s new? IV

Rivière and Tian use this property in a subsequent work on the regularity of J-holomorphic curves. The proof of Rivière is based on a suitable differentiable inequality, which in turn is implied by a suitable modification of an epiperimetric inequality of Brian White.

We state, prove and use several versions of the “splitting-before-tilting” principle in any dimension.

Our proofs are however always based on a perturbation of De Giorgi’s “excess decay”.

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Final blow-up

We can now implement the idea of the plan that failed: assuming the current has too many singularities, show that these singularities are inherited by a suitable (nontrivial) limit of the approximation over the normal bundle of the center manifold, which hopefully is an harmonic Q-valued map.

Singular points are now points where the map essentially collapse on the manifold, which is the "average" of the sheets. So, if the singular points were not inherited in the limit, we would conclude that the order of contact between the current and the center manifold is infinite

Step 4

Recall that the frequency function measures the order of vanishing of an harmonic map.

If we could show that the frequency function of the approximating map is "almost monotone" (and thus bounded), the order of contact with the center manifold would be finite.

In Step 4 we consider the area-functional as a perturbation of the Dirichlet energy to show that the frequency function of the approximating map is almost monotone.

Very important issue: A priori the current is not flat at all scales, i.e. the center manifold might not "go through all scales" up to the singular blow-up point. The frequency function would then not be defined on an interval $]0, r[$, but rather on a sequence of intervals going to 0.

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Even harder to say than with the center manifold.

However, with respect to Almgren we surely take advantage of the monotonicity of the "smoothed" version of the frequency function.

Finally the main reason why the final "blow-up" map inherits the singularities of the current is that we are assuming the latter to be at least $m-2$-dimensional, whereas the converge is of $W^{1,2}$ type (Capacity!).

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Thank you

for your attention!