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ON DOMAIN DECOMPOSITION METHODS FOR OPTIMAL CONTROL PROBLEMS

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Abstract

In this note, we introduce a new approach to study overlapping domain decomposition methods for optimal control systems governed by partial differential equations. The model considered in our paper is systems governed by wave equations. Our technique could be used for several other equations as well.

1. Introduction

The research about using domain decomposition methods to resolve optimal control problems started with the pioneering work of A. Bensoussan, R. Glowinski and P.L. Lions [8] in the 70's and B. Depres and J.D. Benamou in the early 90's [2, 1, 7, 6, 5, 4, 4, 3]. Since then, this research line has become very active with several works of J. E. Lagnese and G. Leugering [13, 11, 10, 9, 12]. However, most of the works on domain decomposition methods for optimal control of systems governed by partial differential equations are devoted to nonoverlapping algorithms, though overlapping algorithms are proved to be more stable and much faster [14]. One of the reasons is that there was no convergence proof of the overlapping algorithms. In the series of papers [17, 16, 18, 15], we develop a new technique to study the convergence of overlapping algorithms. The technique is proved to be applicable for the convergence study of domain decomposition algorithms for several kinds of partial differential equations. Within the frame of developing our new technique for different convergence problems, this note is devoted to the application of the technique to study an overlapping domain decomposition for optimal control systems governed by wave equations, which was studied in [1] but only for the nonoverlapping case. Our technique has the potential of being a new tool to extend many of the previous studies from nonoverlapping to overlapping algorithms. For the sake of simplicity, we only consider a decomposition with two subdomains, however, our technique could be extended to the multisubdomains case without any difficulty.

2. Model description and definition of the domain decomposition algorithm

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Similarly as in [1], we consider the following wave equation defined on $(0,T)\times\Omega$

$$\begin{cases} \partial_{tt} y(t,x) - \Delta y(t,x) = f(t,x) + v(t,x) \text{ on } (0,T)\times\Omega, \\ y(0,x) = y_0(x); \qquad \partial_t y(0,x) = y_1(x) \text{ on } \Omega, \\ y(t,x) = g(t,x) \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{1}$$

where $y_0, y_1 \in L^2(\Omega)$, $g \in L^2((0,T)\times\partial\Omega)$.

Let U be a convex subset of $L^2((0,T)\times\Omega)$ and define the function

$$J(v,y) = \frac{1}{2}\int_{(0,T)\times\Omega} (\gamma|y(x)|^2 + \alpha|v(t,x)|^2)dxdt, \tag{2}$$

where α and γ are positive constants.

We consider the following optimization problem

$$\min_{v\in U} J(v, y(v)). \tag{3}$$

Following [1], we need to solve

$$\begin{cases} \partial_{tt} p\,(t,x) - \Delta p(t,x) = y(t,x) \text{ on } (0,T)\times\Omega, \\ p\,(T,x) = 0; \qquad \partial_t p(T,x) = 0 \text{ on } \Omega, \\ p\,(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \\ \int_{(0,T)\times\Omega} (p+\alpha v)(w-v)dxdt \geq 0 \;\forall w\in U. \end{cases} \tag{4}$$

We now design an overlapping domain decomposition method to resolve the system (1) and (4). Divide the domain Ω into two overlapping subdomains Ω_1 and Ω_2 in the following sense

$$\Omega = \Omega_1 \cup \Omega_2,$$

$$(\partial\Omega_1 \backslash \partial\Omega) \cap (\partial\Omega_2 \backslash \partial\Omega) = \emptyset.$$

The overlapping domain decomposition algorithm with Robin transmission condition now reads for $i \in \{1,2\}$

$$\begin{cases} \partial_{tt} y_i^{n+1} - \Delta y_i^{n+1} = f(t,x) + v_i^{n+1}(t,x) \text{ on } (0,T)\times\Omega_i, \\ y_i^{n+1}(0,x) = y_0(x), \qquad \partial_t y_i^{n+1}(0,x) = y_1(x) \text{ on } \Omega_i, \\ y_i^{n+1}(t,x) = g(t,x) \text{ on } (0,T)\times\partial\Omega_i, \end{cases}$$

$$\begin{cases} \partial_{tt} p_i^{n+1} - \Delta p_i^{n+1} = \gamma y_i^n(t,x) \text{ on } (0,T) \times \Omega_i, \\ p_i^{n+1}(T,x) = 0, \quad \partial_t p_i^{n+1}(T,x) = 0 \text{ on } \Omega_i, \\ p_i^{n+1}(t,x) = 0 \text{ on } (0,T) \times \partial\Omega_i, \end{cases}$$

$$\int_{(0,T)\times\Omega_i} (p_i^{n+1} + \alpha v_i^{n+1})(w_i - v_i^{n+1}) dxdt \geq 0,$$

with the transmission condition on $\partial\Omega_i \backslash \partial\Omega$

$$\partial_{\nu_i} y_i^{n+1} + r_i p_i^{n+1} = \partial_{\nu_i} y_{3-i}^n + r_i p_{3-i}^n,$$
$$\partial_{\nu_i} p_i^{n+1} + r_i y_i^{n+1} = \partial_{\nu_i} p_{3-i}^n + r_i y_{3-i}^n,$$

where ν_i is the outward normal outward unit normal vector of Ω_i on the boundary $\partial\Omega_i \backslash \partial\Omega$ and r_i is a positive constant. At step 0, we choose an initial guess (y_i^0, p_i^0) in $C^2([0,T] \times \overline{\Omega})$. We can see that the algorithm is well-posed and $(y_i^n, p_i^n, v_i^n) \in L^2(0,T,H^2(\Omega_i)) \times L^2(0,T,H^2(\Omega_i)) \times L^2(0,T,H^2(\Omega_i))$.

3. Convergence of the algorithm

For $i \in \{1,2\}$ we define

$$\tilde{y}_i^{n+1} = y_i^{n+1} - y,$$
$$\tilde{p}_i^{n+1} = p_i^{n+1} - p,$$
$$\tilde{v}_i^{n+1} = v_i^{n+1} - v,$$

and get the following systems

$$\begin{cases} \partial_{tt} \tilde{y}_i^{n+1} - \Delta \tilde{y}_i^{n+1} = \tilde{v}_i^{n+1}(t,x) \text{ on } (0,T) \times \Omega_i, \\ \tilde{y}_i^{n+1}(0,x) = 0, \quad \partial_t \tilde{y}_i^{n+1}(0,x) = 0 \text{ on } \Omega_i, \\ \tilde{y}_i^{n+1}(t,x) = 0 \text{ on } (0,T) \times \partial\Omega_i, \end{cases}$$

$$\begin{cases} \partial_{tt} \tilde{p}_i^{n+1} - \Delta \tilde{p}_i^{n+1} = \gamma \tilde{y}_i^n(t,x) \text{ on } (0,T) \times \Omega_i, \\ \tilde{p}_i^{n+1}(T,x) = 0, \quad \partial_t \tilde{p}_i^{n+1}(T,x) = 0 \text{ on } \Omega_i, \\ \tilde{p}_i^{n+1}(t,x) = 0 \text{ on } (0,T) \times \partial\Omega_i, \end{cases}$$

with the transmission condition on $\partial\Omega_i \backslash \partial\Omega$

$$\partial_{\nu_i} \tilde{y}_i^{n+1} + r_i \tilde{p}_i^{n+1} = \partial_{\nu_i} \tilde{y}_{3-i}^n + r_i \tilde{p}_{3-i}^n,$$
$$\partial_{\nu_i} \tilde{p}_i^{n+1} + r_i \tilde{y}_i^{n+1} = \partial_{\nu_i} \tilde{p}_{3-i}^n + r_i \tilde{y}_{3-i}^n.$$

We suppose that for any $n \in \mathbb{N}$, $\tilde{v}_i^n$ is extended by 0 in (T, ∞) and still denote by $\tilde{y}_i^{n+1}$ the solution of

$$\begin{cases} \partial_{tt}\tilde{y}_i^{n+1} - \Delta\tilde{y}_i^{n+1} = \tilde{v}_i^{n+1}(t,x) \text{ on } (0,\infty)\times\Omega_i, \\ \tilde{y}_i^{n+1}(0,x) = 0, \qquad \partial_t\tilde{y}_i^{n+1}(0,x) = 0 \text{ on } \Omega_i, \\ \tilde{y}_i^{n+1}(t,x) = 0 \text{ on } (0,\infty)\times\partial\Omega_i. \end{cases}$$

Using the change of variable $t \to T - t$, we still denote by $\tilde{p}_i^{n+1}$ the solution of

$$\begin{cases} \partial_{tt}\tilde{p}_i^{n+1} - \Delta\tilde{p}_i^{n+1} = \gamma\tilde{y}_i^n(T-t,x) \text{ on } (0,\infty)\times\Omega_i, \\ \tilde{p}_i^n(0,x) = 0, \qquad \partial_t\tilde{p}_i^{n+1}(0,x) = 0 \text{ on } \Omega_i, \\ \tilde{p}_i^{n+1}(t,x) = 0 \text{ on } (0,\infty)\times\partial\Omega_i, \end{cases}$$

with the assumption that $\tilde{y}_i^n(T-t,x) = 0$ for $t > T$. Let H be a positive constant to be chosen later. Define

$$\bar{y}_i^n = \left(\int_0^\infty |\tilde{y}_i^n| \exp\left(-\sqrt{H}t\right) dt\right) g_i^n; \quad \bar{p}_i^n = \left(\int_0^\infty |\tilde{p}_i^n| \exp\left(-\sqrt{H}t\right) dt\right) g_i^n,$$

with $g_i^n \in C^2(\mathbb{R}^N, \mathbb{R})$, $g_i^n > 0$ to be chosen later. For $F : \Omega \to \mathbb{R}$, we define the following norm

$$|||F||| = \left[\int_{\operatorname{supp}(F)} \left|\int_0^\infty |F| \exp\left(-\sqrt{H}t\right) dt\right|^2 dx\right]^{1/2}.$$

Similarly as in [15], a simple calculation leads to

$$-\Delta\bar{y}_i^{n+1} + H\bar{y}_i^{n+1} + \left(-\sum_{\alpha=1}^N \frac{\partial_\alpha g_i^{n+1}}{g_i^{n+1}} + \frac{\nabla g_i^{n+1}}{g_i^{n+1}}\right)\bar{y}_i^{n+1} + \sum_{\alpha=1}^N \frac{2\partial_{x_\alpha} g_i^{n+1}}{g_i}\partial_{x_\alpha}\bar{y}_i^{n+1} \tag{5}$$
$$= \int_0^T v_i^{n+1} \operatorname{sign}(\tilde{y}_i^{n+1}) \exp\left(-\sqrt{H}t\right) dt \text{ on } \Omega_i,$$

$$-\Delta\bar{p}_i^{n+1} + H\bar{p}_i^{n+1} + \left(-2\sum_{\alpha=1}^N \frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^{n+1}} + \frac{\nabla g_i^{n+1}}{g_i^{n+1}}\right)\bar{p}_i^{n+1} + \sum_{\alpha=1}^N 2\frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^n}\partial_{x_\alpha}\bar{p}_i^n \tag{6}$$
$$= \gamma\int_0^T y_i^n(T-t) \operatorname{sign}(\tilde{p}_i^{n+1}) \exp\left(-\sqrt{H}t\right) dt \text{ on } \Omega_i.$$

Choosing g_i^n such that $\nabla g_i^n - r_i g_i^n = 0$ on $\partial\Omega_i \backslash \Omega$, the transmission condition become

$$\begin{aligned}
\partial_{\nu_i} \bar{y}_i^{n+1} =& \partial_{\nu_i} \left(\int_0^\infty |\tilde{y}_i^n| \exp\left(-\sqrt{H}t\right) dt g_i^n \right) \\
=& \left[\int_0^\infty (\partial_{\nu_i} |\tilde{y}_i^n| + r_i |\tilde{y}_i^n|) \exp\left(-\sqrt{H}t\right) dt \right] g_i^n \\
&+ \int_0^\infty |\tilde{y}_i^n| \exp\left(-\sqrt{H}t\right) dt (\partial_{\nu_i} - r_i) g_i^n \\
=& \frac{1}{\beta_i} \partial_{\nu_i} \bar{y}_i^{n+1} \text{ on } \partial\Omega_i \backslash \partial\Omega,
\end{aligned}$$

by choosing g_i^n and g_{3-i}^n, we can make β_i to be a very large positive constant. Similarly, we also have

$$\beta_i \partial_{\nu_i} \bar{p}_i^{n+1} = \partial_{\nu_i} \bar{p}_{3-i}^n.$$

Let φ_{3-i}^n be a function in $H^1(\Omega \backslash \overline{\Omega_i})$ and φ_i^{n+1} be a function in $H^1(\Omega_i)$ such that $\varphi_i^{n+1} = \varphi_{3-i}^n$ on $\partial\Omega_i \backslash \partial\Omega$ and use them as test functions for (5) and (6)

$$\begin{aligned}
&\int_{\Omega \backslash \Omega_i} \nabla \bar{y}_{3-i}^n \nabla \varphi_{3-i}^n dx + \int_{\Omega \backslash \Omega_i} \sum_{\alpha=1}^N 2 \frac{\partial_{x_\alpha} g_{3-i}}{g_{3-i}} \partial_{x_\alpha} \bar{y}_{3-i}^n \varphi_{3-i} dx \\
&+ \int_{\Omega \backslash \Omega_i} \left(\frac{\Delta g_{3-i}}{g_{3-i}} - 2 \sum_{\alpha=1}^N \frac{\partial_{x_\alpha} g_{3-i}}{g_{3-i}} \right) \bar{y}_{3-i}^n \varphi_{3-i}^n dx + \int_{\Omega \backslash \Omega_i} H \bar{y}_{3-i}^n \varphi_{3-i}^n dx \\
&- \int_{\Omega \backslash \Omega_i} \int_0^T v_{3-i}^n \text{sign}\,(\tilde{y}_{3-i}^n) \exp\left(-\sqrt{H}t\right) dt \varphi_{3-i}^n dx \\
=& - \beta_i \Bigg\{ \int_{\Omega_i} \nabla \bar{y}_i^{n+1} \nabla \varphi_i^{n+1} dx + \int_{\Omega_i} \sum_{\alpha=1}^N 2 \frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^{n+1}} \partial_{x_\alpha} \bar{y}_i^{n+1} \varphi_i^{n+1} dx \qquad (7) \\
&+ \int_{\Omega_i} \left(\frac{\Delta g_i^{n+1}}{g_i^{n+1}} - 2 \sum_{\alpha=1}^N \frac{\partial_{x_\alpha} g_i^{n+1}}{g_i^{n+1}} \right) \bar{y}_i^{n+1} \varphi_i^{n+1} dx + \int_{\bar{\Omega}_i} H \bar{y}_i^{n+1} \varphi_i^{n+1} dx \\
&- \int_{\Omega_i} \int_0^T v_i^{n+1} \text{sign}\,(\tilde{y}_i^{n+1}) \exp\left(-\sqrt{H}t\right) dt \varphi_i^{n+1} dx \Bigg\}.
\end{aligned}$$

In the above equation choose φ_i^{n+1} to be $\bar{y}_i^{n+1}$. Then there exists a function ρ such that ρ is defined on $\Omega \backslash \Omega_i$ and

$$\begin{aligned}
\|\rho\|_{H^1(\Omega \backslash \Omega_i)} \leq C_1 \|\bar{y}_i^{n+1}\|_{H^1(\Omega_i)}, \\
\|\rho\|_{L^2(\Omega \backslash \Omega_i)} \leq C_1 \|\bar{y}_i^{n+1}\|_{L^2(\Omega_i)},
\end{aligned}$$

where C_1 is a positive constant depending on Ω, Ω_1, and Ω_2. Choose φ_{3-i}^n to be ρ, then for H large enough, (7) implies

$$\begin{aligned}
&\sum_{i=1}^{2} C_2 \Bigg\{ \frac{1}{2} \int_{\Omega\setminus\Omega_i} |\nabla \bar{y}_{3-i}^n|^2 dx + \frac{H}{2} \int_{\Omega\setminus\Omega_i} |\bar{y}_{3-i}^n|^2 dx \\
&- \int_{\Omega\setminus\Omega_i} \int_0^T v_{3-i}^n \mathrm{sign}\,(\tilde{y}_{3-i}^n) \exp\left(-\sqrt{H}t\right)\, dt \bar{y}_{3-i}^n dx \Bigg\} \\
\geq &\sum_{i=1}^{2} \beta_i \Bigg\{ \frac{1}{2} \int_{\Omega_i} |\nabla \bar{y}_i^{n+1}|^2 dx + \frac{H}{2} \int_{\Omega_i} |\bar{y}_i^{n+1}|^2 dx \\
&- \int_{\Omega_i} \int_0^T v_i^{n+1} \mathrm{sign}\,(\tilde{y}_i^{n+1}) \exp\left(-\sqrt{H}t\right)\, dt \bar{y}_i^{n+1} dx \Bigg\},
\end{aligned} \tag{8}$$

where C_2 is some constants depending only on the structure of the equation. In a similar way, we have

$$\begin{aligned}
&\sum_{i=1}^{2} C_3 \Bigg\{ \frac{1}{2} \int_{\Omega\setminus\Omega_i} |\nabla \bar{p}_{3-i}^n|^2 dx + \frac{H}{2} \int_{\Omega\setminus\Omega_i} |\bar{p}_{3-i}^n|^2 dx \\
&- \gamma \int_{\Omega\setminus\Omega_i} \int_0^T y_{3-i}^{n-1} \mathrm{sign}\,(\tilde{p}_{3-i}^n) \exp\left(-\sqrt{H}t\right)\, dt \bar{p}_{3-i}^n dx \Bigg\} \\
\geq &\sum_{i=1}^{2} \beta_i \Bigg\{ \frac{1}{2} \int_{\Omega_i} |\nabla \bar{p}_i^{n+1}|^2 dx + \frac{H}{2} \int_{\Omega_i} |\bar{p}_i^{n+1}|^2 dx \\
&- \gamma \int_{\Omega_i} \int_0^T y_i^n \mathrm{sign}\,(\tilde{p}_i^{n+1}) \exp\left(-\sqrt{H}t\right)\, dt \phi_i^{n+1} dx \Bigg\},
\end{aligned}$$

where ϕ_i^{n+1} plays a similar role as the role of ϕ_i^{n+1} in the estimate of $\bar{y}_i^{n+1}$

$$\begin{aligned}
\|\phi_i^{n+1}\|_{H^1(\Omega\setminus\Omega_i)} &\leq C_1 \|\bar{p}_i^{n+1}\|_{H^1(\Omega_i)}, \\
\|\phi_i^{n+1}\|_{L^2(\Omega\setminus\Omega_i)} &\leq C_1 \|\bar{p}_i^{n+1}\|_{L^2(\Omega_i)}.
\end{aligned}$$

Similarly as [15], taking β_i and H to be very large, and using the equation (as in [1])

$$\int_{(0,T)\times\Omega_i} (p_i^{n+1} + \alpha v_i^{n+1})(w_i - v_i^{n+1})\, dxdt \geq 0,$$

we get

$$\lim_{n\to\infty} (|||\nabla y_i^n||| + |||y_i^n||| + |||\nabla p_i^n||| + |||p_i^n|||) = 0.$$

Notice that the fact $|||\nabla y_i^n|||$, $|||y_i^n|||$, $|||\nabla p_i^n|||$, $|||p_i^n|||$, $|||v_i^n|||$ are well-defined is also included in the convergence result.

Theorem 3.1 *The algorithm converges in the following sense:*

$$\lim_{n\to\infty}(|||\nabla y_i^n||| + |||y_i^n||| + |||\nabla p_i^n||| + |||p_i^n||| + |||v_i^n|||) = 0.$$

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References

[1] Benamou, J.-D.: Domain decomposition, optimal control of systems governed by partial differential equations, and synthesis of feedback laws. J. Optim. Theory Appl. **102**(1) (1999), 15–36.

[2] Benamou, J.-D. and Desprès, B.: A domain decomposition method for the Helmholtz equation and related optimal control problems. J. Comput. Phys. **136**(1) (1997), 68–82.

[3] Benamou, J.-D.: Décomposition de domaine pour le réarrangement monotone d'applications vectorielles. C. R. Acad. Sci. Paris Sér. I Math. **315**(4) (1992), 469–474.

[4] Benamou, J.-D.: Décomposition de domaine pour le contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles elliptiques. C. R. Acad. Sci. Paris Sér. I Math. **317**(2) (1993) 205–209.

[5] Benamou, J.-D.: A domain decomposition method for the optimal control of systems governed by the Helmholtz equation. In: *Mathematical and numerical aspects of wave propagation (Mandelieu-La Napoule, 1995)*, pp. 653–662. SIAM, Philadelphia, PA, 1995.

[6] Benamou, J.-D.: A domain decomposition method with coupled transmission conditions for the optimal control of systems governed by elliptic partial differential equations. SIAM J. Numer. Anal. **33**(6) (1996), 2401–2416.

[7] Benamou, J.-D.: Décomposition de domaine pour le contrôle de systèmes gouvernés par des équations d'évolution. C. R. Acad. Sci. Paris Sér. I Math. **324**(9) (1997), 1065–1070.

[8] Bensoussan, A., Glowinski, R., and Lions, J.-L.: Méthode de décomposition appliquée au contrôle optimal de systèmes distribués. In: *Fifth Conferences on Optimization Techniques (Rome, 1973), Part I, Lecture Notes in Comput. Sci.*, vol. 3, pp. 141–151. Springer, Berlin, 1973.

[9] Lagnese, J. E. and Leugering, G.: Time-domain decomposition of optimal control problems for the wave equation. Systems Control Lett. **48**(3–4) (2003), 229–242. Optimization and control of distributed systems.

[10] Lagnese, J. E. and Leugering, G.: Domain decomposition methods in optimal control of partial differential equations. *International Series of Numerical Mathematics*, vol. 148, Birkhäuser Verlag, Basel, 2004.

[11] Leugering, G.: Domain decomposition in optimal control problems for partial differential equations revisited. In: *Control theory of partial differential equations*, *Lect. Notes Pure Appl. Math.*, vol. 242, pp. 125–155. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[12] Leugering, G.: Domain decomposition of optimal control problems for dynamic networks of elastic strings. Comput. Optim. Appl. **16**(1) (2000), 5–27.

[13] Leugering, G.: Domain decomposition of constrained optimal control problems for 2D elliptic system on networked domains: convergence and a posteriori error estimates. In: *Domain decomposition methods in science and engineering XVII*, *Lect. Notes Comput. Sci. Eng.*, vol. 60, pp. 119–130. Springer, Berlin, 2008.

[14] Toselli, A. and Widlund, O.: Domain decomposition methods – algorithms and theory. *Springer Series in Computational Mathematics*, vol. 34. Springer-Verlag, Berlin, 2005.

[15] Tran, M.-B.: Optimized overlapping domain decomposition: Convergence proofs. *Domain Decomposition Methods in Science and Engineering XX; Series: Lecture Notes in Computational Science and Engineering, Springer*. To appear.

[16] Tran, M.-B. Overlapping optimized Schwarz methods for parabolic equations in n-dimensions. *Proceedings of the American Mathematical Society*. To appear.

[17] Tran, M.-B. Parallel Schwarz waveform relaxation method for a semilinear heat equation in a cylindrical domain. C. R. Math. Acad. Sci. Paris **348**(13-14) (2010), 795–799.

[18] Tran, M.-B.: A parallel four step domain decomposition scheme for coupled forward-backward stochastic differential equations. J. Math. Pures Appl. (9) **96**(4) (2011), 377–394.