Conference Applications of Mathematics 2013
in honor of the 70th birthday of Karel Segeth.
Jan Brandts, Sergey Korotov, Michal Křížek,
Jakub Šístek, and Tomáš Vejchodský (Eds.),
Institute of Mathematics AS CR, Prague 2013

ZERO POINTS OF QUADRATIC MATRIX POLYNOMIALS

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Abstract

Our aim is to classify and compute zeros of the quadratic two sided matrix polynomials, i.e. quadratic polynomials whose matrix coefficients are located at both sides of the powers of the matrix variable. We suppose that there are no multiple terms of the same degree in the polynomial $\mathbf{p}$, i.e., the terms have the form $\mathbf{A}_j\mathbf{X}^j\mathbf{B}_j$, where all quantities $\mathbf{X}, \mathbf{A}_j, \mathbf{B}_j, j = 0, 1, \ldots, N$, are square matrices of the same size. Both for classification and computation, the essential tool is the description of the polynomial $\mathbf{p}$ by a matrix equation $\mathbf{P}(\mathbf{X}) := \mathbf{A}(\mathbf{X})\mathbf{X} + \mathbf{B}(\mathbf{X})$, where $\mathbf{A}(\mathbf{X})$ is determined by the coefficients of the given polynomial $\mathbf{p}$ and $\mathbf{P},\ \mathbf{X},\ \mathbf{B}$ are real column vectors. This representation allows us to classify five types of zero points of the polynomial $\mathbf{p}$ in dependence on the rank of the matrix $\mathbf{A}$. This information can be for example used for finding all zeros in the same class of equivalence if only one zero in that class is known. For computation of zeros, we apply Newtons method to $\mathbf{P}(\mathbf{X}) = \mathbf{0}$.

1. Introduction

In papers [4, 5] we have investigated quaternionic polynomials of the one-sided and the two-sided type. The one-sided type is described by terms of the form a_jx^j or x^ja_j, whereas the two-sided type is described by terms of the form $a_jx^jb_j,\ j \geq 0$. In this paper we will consider matrix polynomials which have matrix coefficients and a matrix variable as well, i.e. the terms have the form $\mathbf{A}_j\mathbf{X}^j\mathbf{B}_j$. All quantities $\mathbf{X}$, $\mathbf{A}_j$, $\mathbf{B}_j$, $j = 0, 1, \ldots, N$, are square matrices of the same size.

We will use the notation $\mathbb{R}, \mathbb{C}$ for the field of real and complex numbers, respectively; $\mathbb{K}$ will stand for $\mathbb{R}$ or $\mathbb{C}$. The set of square matrices over $\mathbb{K}$ will be denoted by $\mathbb{K}^{n\times n}$, where n is the order of the matrix. By $\mathbf{I} \in \mathbb{K}^{n\times n}$ we will denote the identity matrix, the matrix $\mathbf{0} \in \mathbb{K}^{n\times n}$ is the zero matrix.

Since the general task is very complicated, in this paper we will restrict ourselves to quadratic matrix polynomials without multiple terms of the same degree: for given $\mathbf{A}_0,\ \mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{B}_0,\ \mathbf{B}_1,\ \mathbf{B}_2 \in \mathbb{K}^{n\times n}$, we consider quadratic polynomial $\mathbf{p}$ in the form

$$\mathbf{p}(\mathbf{X}) = \mathbf{A}_0\mathbf{B}_0 + \mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{A}_2\mathbf{X}^2\mathbf{B}_2, \quad \text{where} \quad \mathbf{A}_0\mathbf{B}_0,\, \mathbf{A}_2,\, \mathbf{B}_2 \neq \mathbf{0}\,. \tag{1}$$

The condition $\mathbf{A}_0\mathbf{B}_0 \neq \mathbf{0}$ implies that $\mathbf{p}(\mathbf{0}) \neq \mathbf{0}$. The conditions $\mathbf{A}_2,\ \mathbf{B}_2 \neq \mathbf{0}$ imply that the term with the degree 2 is nonvanishing.

If the matrix $\mathbf{X}$ has the property $\mathbf{p}(\mathbf{X}) = \mathbf{0}$, we will call $\mathbf{X}$ a zero of $\mathbf{p}$.

As an example, let us consider matrices of the order $n = 2$. In this case the quadratic matrix polynomial can be formally transformed into a linear system of four equations (for $n = 2$, it is true for polynomials of any degree N) and we will classify the zeros of the polynomial in terms of the rank of the corresponding system.

In general, we transform the quadratic matrix polynomial $\mathbf{p}$ into a matrix equation $\mathbf{P}(\mathbf{X}) := \mathbf{A}(\mathbf{X})\mathbf{X} + \mathbf{B}(\mathbf{X})$, where $\mathbf{A}(\mathbf{X})$ is determined by the coefficients of the given polynomial $\mathbf{p}$ and $\mathbf{P},\ \mathbf{X},\ \mathbf{B}$ are real column vectors. Then we classify zeros by the rank of the matrix $\mathbf{A}$. We showed that in general there are five different types of zeros.

For computation of zeros, we apply Newton's method to the matrix equation $\mathbf{P}(\mathbf{X}) = \mathbf{0}$.

2. Preliminaries

This section contains basic facts from the theory of matrices. It can be found e. g. in Horn and Johnson, [2].

Let $\mathbb{A} \in \mathbb{K}^{n\times n}$. Then $\chi_{\mathbf{A}}(z) := \det(z\mathbf{I} - \mathbf{A}) = z^n + a_{n-1}^{(n)} z^{n-1} + \cdots + a_0^{(n)}$ is called the characteristic polynomial of $\mathbf{A}$. Cayley–Hamilton theorem says that the matrix $\mathbf{A}$ annihilates its characteristic polynomial,

$$\chi_{\mathbf{A}}(\mathbf{A}) = \mathbf{A}^n + \cdots + a_0^{(n)}\mathbf{I} = \mathbf{0}\,. \tag{2}$$

In particular, for $n = 2$ we have

$$\mathbf{A}^2 - \operatorname{tr}(\mathbf{A})\mathbf{A} + \det(\mathbf{A})\mathbf{I} = \mathbf{0}\,.$$

Let us recall that two matrices $\mathbf{A}$, $\mathbf{B}$ of the same order over $\mathbb{K}$ are similar if there is a nonsingular matrix $\mathbf{H}$ of the same order such that $\mathbf{A} = \mathbf{H}\mathbf{B}\mathbf{H}^{-1}$.

For fixed $\mathbf{A} \in \mathbb{K}^{n\times n}$ the set of matrices

$$[\mathbf{A}] = \{\,\mathbf{B}, \mathbf{B} = \mathbf{H}\mathbf{A}\mathbf{H}^{-1} \text{ for all nonsingular } \mathbf{H}\} \tag{3}$$

is called similarity class of $\mathbf{A}$. The similarity class is finite only for multiples of the identity matrix: if $\mathbf{A} = c\mathbf{I},\ c \in \mathbb{K}$, then $[\mathbf{A}] = \{\mathbf{A}\}$ consists only of one element.

There are two special cases of (1) worth mentioning. If we put $\mathbf{X} := z\mathbf{I} \in \mathbb{K}^{n\times n}$, where $z \in \mathbb{K}$, we obtain

$$\mathbf{p}(\mathbf{X}) = \mathbf{p}(z\mathbf{I}) = \mathbf{C}_0 + \mathbf{C}_1\, z + \mathbf{C}_2\, z^2, \quad \mathbf{C}_j = \mathbf{A}_j\mathbf{B}_j,\ j = 0, 1, 2\,. \tag{4}$$

If all coefficients have the special form $\mathbf{A}_j = \alpha_j\mathbf{I} \in \mathbb{K}^{n\times n}$, $\mathbf{B}_j = \beta_j\mathbf{I} \in \mathbb{K}^{n\times n}$, $\gamma_j := \alpha_j\beta_j$, $j = 0, 1, 2$, we obtain

$$\mathbf{p}(\mathbf{X}) = \gamma_0\mathbf{I} + \gamma_1\mathbf{X} + \gamma_2\mathbf{X}^2\,. \tag{5}$$

Both forms have their ranges in $\mathbb{K}^{n\times n}$, see also [7, 3].

Definition The set of matrices

$$\mathcal{C} := \{\mathbf{M} :\ \mathbf{M} = a\mathbf{I} \in \mathbb{K}^{n\times n}\} \tag{6}$$

is called the *center* of $\mathbb{K}^{n\times n}$.

Remark In general terms the center of a noncommutative (semi)group $\mathcal{G}$ is the set of all elements, which commute with all elements of $\mathcal{G}$.

If we want to find out whether an element of the center $\mathcal{C}$ is a zero of a given quadratic matrix polynomial $\mathbf{p}$, then, we have to use the form (4), namely

$$\mathbf{p}(z\mathbf{I}) = \mathbf{C}_0 + \mathbf{C}_1 z + \mathbf{C}_2 z^2 = \mathbf{0} \in \mathbb{K}^{n\times n}, \quad \mathbf{C}_j = \mathbf{A}_j\mathbf{B}_j,\ j = 0, 1, 2\,. \tag{7}$$

This matrix equation separates into n^2 standard polynomial equations: Let $\mathbf{C}_j := (c_{kl}^{(j)})$, $k, l = 1, 2, \ldots, n$, $j = 0, 1, 2$. Then (7) is equivalent to a system of n^2 equations

$$c_{kl}^{(0)} + c_{kl}^{(1)} z + c_{kl}^{(2)} z^2 = 0, \quad k, l = 1, 2, \ldots, n\,. \tag{8}$$

This allows us to assume, that in the sequel we are looking only for solutions $\mathbf{X} \notin \mathcal{C}$.

Lemma Let $\mathbf{p}$ be a quadratic polynomial defined by the coefficients $\mathbf{A}_i, \mathbf{B}_i \in \mathbb{K}^{n\times n}$, $i = 0, 1, 2$, and let $\mathbf{q}$ be a quadratic polynomial defined by the coefficients $\mathbf{H}^{-1}\mathbf{A}_i\mathbf{H}$, $\mathbf{H}^{-1}\mathbf{B}_i\mathbf{H}$, $i = 0, 1, 2$, for a fixed nonsingular matrix $\mathbf{H} \in \mathbb{K}^{n\times n}$. Then,

$$\mathbf{p}(\mathbf{X}) = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{q}(\mathbf{H}^{-1}\mathbf{X}\mathbf{H}) = \mathbf{0}. \tag{9}$$

Proof For the quadratic polynomial $\mathbf{q}$, we have

$$\begin{aligned}
\mathbf{q}(\mathbf{X}) =& (\mathbf{H}^{-1}\mathbf{A}_0\mathbf{H})\mathbf{X}^0(\mathbf{H}^{-1}\mathbf{B}_0\mathbf{H}) + \\
& + (\mathbf{H}^{-1}\mathbf{A}_1\mathbf{H})\mathbf{X}^1(\mathbf{H}^{-1}\mathbf{B}_1\mathbf{H}) + (\mathbf{H}^{-1}\mathbf{A}_2\mathbf{H})\mathbf{X}^2(\mathbf{H}^{-1}\mathbf{B}_2\mathbf{H}) = \\
=& \mathbf{H}^{-1}\left(\mathbf{A}_0(\mathbf{H}\mathbf{X}^0\mathbf{H}^{-1})\mathbf{B}_0 + \mathbf{A}_1(\mathbf{H}\mathbf{X}^1\mathbf{H}^{-1})\mathbf{B}_1 + \mathbf{A}_2(\mathbf{H}\mathbf{X}\mathbf{H}^{-1})^2\mathbf{B}_2\right)\mathbf{H} = \\
=& \mathbf{H}^{-1}\mathbf{p}(\mathbf{H}\mathbf{X}\mathbf{H}^{-1})\mathbf{H},
\end{aligned}$$

which implies that $\mathbf{q}(\mathbf{H}^{-1}\mathbf{X}\mathbf{H}) = \mathbf{H}^{-1}\mathbf{p}(\mathbf{X})\mathbf{H}$. Or in other words $\mathbf{p}(\mathbf{X})$ is similar to $\mathbf{q}(\mathbf{H}^{-1}\mathbf{X}\mathbf{H})$ and (9) follows.

3. Quadratic matrix polynomial of order two

Let us assume that all occurring matrices have the order $n = 2$.

The following recursion was for the first time used by Horn and Johnson, see [2].

Theorem Let $\mathbf{X} \in \mathbb{K}^{2\times 2}$ and let $\chi_{\mathbf{X}}(z) := z^2 - \mathrm{tr}(\mathbf{X})z + \det(\mathbf{X})$ be its characteristic polynomial. Then, there are numbers $\alpha_j,\ \beta_j,\ j \geq 0$, such that

$$\mathbf{X}^j = \alpha_j \mathbf{X} + \beta_j \mathbf{I} \text{ for all } j = 0, 1, \ldots, \tag{10}$$

where

$$\begin{aligned} \alpha_0 &:= 0, \quad \beta_0 := 1, \\ \alpha_{j+1} &:= \mathrm{tr}(\mathbf{X})\alpha_j + \beta_j, \\ \beta_{j+1} &:= -\alpha_j \det(\mathbf{X}), \quad j \geq 0. \end{aligned}$$

In particular,

$$\begin{aligned} \alpha_1 &:= 1, & \beta_1 &:= 0, \\ \alpha_2 &:= \mathrm{tr}(\mathbf{X}), & \beta_2 &:= -\det(\mathbf{X}). \end{aligned}$$

If the coefficients of the characteristic polynomial are real, then also all $\alpha_j,\ \beta_j$ are real for all j.

Proof From the Cayley–Hamilton theorem we have

$$\mathbf{X}^2 = \mathrm{tr}(\mathbf{X})\mathbf{X} - \det(\mathbf{X})\mathbf{I}\,. \tag{11}$$

If we multiply (10) by $\mathbf{X}$ and replace $\mathbf{X}^2$ with the right-hand side of the equation (11), we obtain

$$\begin{aligned} \mathbf{X}^{j+1} &= \alpha_j(\mathrm{tr}(\mathbf{X})\mathbf{X} - \det(\mathbf{X})\mathbf{I}) + \beta_j\mathbf{X} = (\alpha_j \mathrm{tr}(\mathbf{X}) + \beta_j)\mathbf{X} - \alpha_j \det(\mathbf{X})\mathbf{I} = \\ &= \alpha_{j+1}\mathbf{X} + \beta_{j+1}\mathbf{I}\,, \end{aligned}$$

from which the desired recursion in (10) follows. □

The theorem says that a power $\mathbf{X}^j$, $j = 0, 1, \ldots,$ of a matrix $\mathbf{X}$ of order 2, regardless of the power j, can always be expressed as a linear combination of the matrix $\mathbf{X}$ and the identity matrix $\mathbf{I}$.

Remark In general, for a matrix $\mathbf{X}$ of order n a power $\mathbf{X}^j$ can always be expressed as an element of the linear hull of matrices $\mathbf{X}^{\nu-1},\ \mathbf{X}^{\nu-2}, \ldots, \mathbf{I}$, where ν is the degree of the minimal polynomial of $\mathbf{X}$, see [2].

Remark The corresponding iteration given by Pogurui and Shapiro in [9] is three term recursion, whereas (10) is a two term recursion. Formally, they differ. In some cases, two term recursions are more stable than the corresponding three term recursions. For an example, see [8].

We apply formula (11). Then our quadratic polynomial $\mathbf{p}(\mathbf{X})$ in (1) has the form

$$\mathbf{p}(\mathbf{X}) = \mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathrm{tr}(\mathbf{A})\mathbf{A}_2\mathbf{X}\mathbf{B}_2 + \mathbf{A}_0\mathbf{B}_0 - \det(\mathbf{X})\mathbf{A}_2\mathbf{B}_2\,. \tag{12}$$

Now, let $n \geq 2$ and let $\mathbf{X} \in \mathbb{K}^{n\times n}$, $\mathbf{X} := (x_{j,k}),\ j,k = 1,2,\ldots,n$. We define the operator

$$\mathrm{col} : \mathbb{K}^{n\times n} \to \mathbb{K}^{n^2\times 1}\,,$$

$$\mathrm{col}(\mathbf{X}) := \left(x_{11}, x_{21}, \cdots, x_{n1}, x_{12}, x_{22}, \cdots, x_{n2}, \cdots, x_{1n}, x_{2n}, \cdots, x_{nn}\right)^{\mathrm{T}}\,.$$

In particular for $\mathbf{X} \in \mathbb{K}^{2\times 2}$,

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \text{we have} \quad \mathrm{col}(\mathbf{X}) := \left(x_{11}, x_{21}, x_{12}, x_{22}\right)^{\mathrm{T}}\,.$$

Let us note that col is an invertible linear mapping, $\mathrm{col} : \mathbb{K}^{n\times n} \longrightarrow \mathbb{K}^{n^2}$.

Let $\mathbf{A}, \mathbf{B}, \mathbf{X} \in \mathbb{K}^{n\times n}$. Let f be a linear mapping, $f : \mathbb{K}^{n\times n} \longrightarrow \mathbb{K}^{n\times n}$, defined as

$$f(\mathbf{X}) = \mathbf{AXB}\,, \tag{13}$$

represented by the Kronecker product in the form

$$\mathrm{col}(f(\mathbf{X})) = (\mathbf{B}^{\mathrm{T}} \otimes \mathbf{A})\mathrm{col}(\mathbf{X}). \tag{14}$$

Applying col to (12) and using (14), we obtain, see also [1],

$$\mathbf{P}(\mathbf{X}) := \mathrm{col}(\mathbf{p}(\mathbf{X})) = \mathbf{M}(\mathbf{X})\mathrm{col}(\mathbf{X}) + \mathbf{N}(\mathbf{X}), \tag{15}$$

where

$$\mathbf{M}(\mathbf{X}) = (\mathbf{B}_1^{\mathrm{T}} \otimes \mathbf{A}_1) + \mathrm{tr}(\mathbf{X})(\mathbf{B}_2^{\mathrm{T}} \otimes \mathbf{A}_2)\,, \tag{16}$$

$$\mathbf{N}(\mathbf{X}) = \mathrm{col}(\mathbf{A}_0\mathbf{B}_0 - \det(\mathbf{X})\mathbf{A}_2\mathbf{B}_2). \tag{17}$$

Let us remark that both $\mathbf{M}(\mathbf{X})$ and $\mathbf{N}(\mathbf{X})$ depend on $\mathbf{X}$ or more precisely on $\mathrm{tr}(\mathbf{X})$ and $\det(\mathbf{X})$. This means, that the matrices $\mathbf{M}(\mathbf{X})$ and $\mathbf{N}(\mathbf{X})$ are constant on the equivalence class $[\mathbf{X}]$.

Corollary Let $\mathbf{P}(\mathbf{X}) := \mathbf{M}(\mathbf{X})\mathrm{col}(\mathbf{X}) + \mathbf{N}(\mathbf{X}) = \mathbf{0}$. Then all (further) zeros $\mathbf{Y}$ of $\mathbf{P}$ in $[\mathbf{X}]$ can be determined by solving the linear 4×4 system

$$\mathbf{M}(\mathbf{X})\mathrm{col}(\mathbf{Y}) + \mathbf{N}(\mathbf{X}) = \mathbf{0}. \tag{18}$$

If the matrix $\mathbf{M}$ is nonsingular (we delete the arguments), then there is only one zero of $\mathbf{P}$ in $[\mathbf{X}]$. If the matrix $\mathbf{M}$ is the zero matrix, then $\mathbf{N} = \mathbf{0}$ and all matrices in $[\mathbf{X}]$ are zeros of $\mathbf{P}$. If $\mathbf{N} = \mathbf{0}$, then $\mathbf{M}$ is singular.

Since the zeros of $\mathbf{P}$ are eventually all solutions of the linear system (18), we can classify them according to the rank of $\mathbf{M}(\mathbf{X})$.

Definition Let $\mathbf{P}(\mathbf{X}) := \mathbf{M}(\mathbf{X})\mathrm{col}(\mathbf{X}) + \mathbf{N}(\mathbf{X}) = \mathbf{0}$ and let $\mathbf{X} \neq a\mathbf{I},\ a \in \mathbb{R}$. We say that $\mathbf{X}$ is a zero of rank k if $\mathrm{rank}(\mathbf{M}(\mathbf{X})) = k,\ 0 \leq k \leq 4$. A zero of rank 0 will be called spherical zero, a zero of rank 4 will be called isolated zero. If $\mathbf{X} = a\mathbf{I},\ a \in \mathbb{R}$, the zero will also be called isolated.

Remark In [5], we have shown that for quaternionic polynomials zeros of all ranks, zero to four, exist. For the geometrical meaning of the term "spherical zeros" see [10].

As an example, let us have a special quadratic polynomial

$$\mathbf{p}(\mathbf{X}) := \mathbf{X}^2 + \alpha_1\mathbf{X} + \alpha_0\mathbf{I}, \quad \alpha_1,\ \alpha_0 \in \mathbb{K},\ \alpha_0 \neq 0, \quad \mathbf{X} \in \mathbb{K}^{2\times 2}, \tag{19}$$

which according to (12) can also be written as

$$\mathbf{P}(\mathbf{X}) = (\alpha_1 + \mathrm{tr}(\mathbf{X}))\mathrm{col}(\mathbf{X}) + (\alpha_0 - \det(\mathbf{X}))\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

or equivalently $\mathbf{p}(\mathbf{X}) = (\alpha_1 + \mathrm{tr}(\mathbf{X}))\mathbf{X} + (\alpha_0 - \det(\mathbf{X}))\mathbf{I}$.

Then, there are two cases for all zeros $\mathbf{X}$ of $\mathbf{p}$:

1. $\alpha_1 + \mathrm{tr}(\mathbf{X}) = \alpha_0 - \det(\mathbf{X}) = 0,$
2. $\alpha_1 + \mathrm{tr}(\mathbf{X}) \neq 0,\ \alpha_0 - \det(\mathbf{X}) \neq 0.$

All matrices which are not a real multiple of the identity matrix $\mathbf{I}$ and obey the equations of the first case are spherical zeros of the given polynomial, they form an equivalence class of spherical zeros. And there are no other spherical zeros. Put

$$\mathbf{X} := \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}. \tag{20}$$

Then all spherical solutions have the form

$$\mathbf{X} := \begin{pmatrix} -\alpha_1 - x_4 & x_3 \\ x_2 & x_4 \end{pmatrix},$$

where x_2, x_3 are arbitrary and

$$x_4 := -\frac{1}{2}\left(\alpha_1 \pm \sqrt{\alpha_1^2 - 4(\alpha_0 + x_2x_3)}\right).$$

Let the second case be valid. In this case, there may exist other zeros than spherical ones, which are of rank four and which must have the form

$$\mathbf{X} = -\frac{\alpha_0 - \det(\mathbf{X})}{\alpha_1 + \mathrm{tr}(\mathbf{X})}\mathbf{I} =: a\mathbf{I}.$$

Since $\det(\mathbf{X}) = a^2,\ \mathrm{tr}(\mathbf{X}) = 2a$, we obtain

$$a := \frac{1}{2}\left(-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0}\right).$$

To summarize: Matrix polynomials (19) have always one spherical zero and in addition two isolated zeros (if $\alpha_1^2 - 4\alpha_0 \neq 0$) or one isolated zero (if $\alpha_1^2 - 4\alpha_0 = 0$). All in all, $\mathbf{p}$ has two or three zeros.

Example Consider the following quadratic polynomial with matrices of order $n = 2$:

$$\mathbf{p}(\mathbf{X}) := \mathbf{X}^2 - \mathbf{X} - \mathbf{I}, \tag{21}$$

i.e.

$$\alpha_1 = -1, \quad \alpha_0 = -1, \quad \alpha_1^2 - 4\alpha_0 = 5 \neq 0\,. \tag{22}$$

The matrix polynomial (21) has two isolated zeros

$$\mathbf{X}_1 = \frac{1}{2}\begin{pmatrix} 1+\sqrt{5} & 0 \\ 0 & 1+\sqrt{5} \end{pmatrix}, \qquad \mathbf{X}_2 = \frac{1}{2}\begin{pmatrix} 1-\sqrt{5} & 0 \\ 0 & 1-\sqrt{5} \end{pmatrix}$$

and there is also one spherical zero

$$\mathbf{X}_3 = \begin{pmatrix} 1-x_4 & x_3 \\ x_2 & x_4 \end{pmatrix},$$

where $x_4 = \dfrac{1}{2}(1 \pm \sqrt{5 - 4x_2x_3})$, x_2, x_3 arbitrary. Let us put, e. g., $x_2 = x_3 = 0$. We obtain

$$x_4^+ = \frac{1}{2}(1+\sqrt{5}), \qquad x_4^- = \frac{1}{2}(1-\sqrt{5})\,.$$

Accordingly, for the spherical root $\mathbf{X}_3$ we have

$$\mathbf{X}_3^+ = \frac{1}{2}\begin{pmatrix} 1-\sqrt{5} & 0 \\ 0 & 1+\sqrt{5} \end{pmatrix}, \qquad \mathbf{X}_3^- = \frac{1}{2}\begin{pmatrix} 1+\sqrt{5} & 0 \\ 0 & 1-\sqrt{5} \end{pmatrix}.$$

It is an easy exercise to show that $\mathbf{X}_3^+$ and $\mathbf{X}_3^-$ belong to the same equivalence class:

$$\mathbf{P}\mathbf{X}_3^+\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1-\sqrt{5} & 0 \\ 0 & 1+\sqrt{5} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{X}_3^-\,.$$

Thus the polynomial $\mathbf{p}$ of (21) has altogether three zeros, one spherical and two isolated ones.

Lemma In order that the quadratic polynomial $\mathbf{p}$, defined in (12), has a spherical zero, it is necessary that

$$(\mathbf{B}_1^{\mathrm{T}} \otimes \mathbf{A}_1) = -\mathrm{tr}(\mathbf{X})(\mathbf{B}_2^{\mathrm{T}} \otimes \mathbf{A}_2) \quad \text{and} \quad \mathbf{A}_0\mathbf{B}_0 = -\det(\mathbf{X})\mathbf{A}_2\mathbf{B}_2\,.$$

Proof It follows directly from the definition of spherical zeros. □

Corollary Let $\mathbf{A}, \mathbf{B}$ be arbitrary nonvanishing matrices in $\mathbb{K}^{2\times 2}$. A necessary condition for spherical zeros to exist is that p has the form

$$\mathbf{p}(\mathbf{X}) := \mathbf{A}\mathbf{X}^2\mathbf{B} + \alpha_1\mathbf{A}\mathbf{X}\mathbf{B} + \alpha_0\mathbf{A}\mathbf{B}, \quad \mathbf{A}\mathbf{B} \neq \mathbf{0}, \tag{23}$$

for certain α_0, α_1.

On the other hand, not for each choice of α_0, α_1 does this lead to spherical zeros.

Remark Polynomials with order two matrices of any degree could be treated in a similar way as we did it here.

4. Numerical considerations for finding the zeros

Let us restrict ourselves to quadratic matrix polynomials with $n = 2$.

We apply Newton's method to

$$\mathbf{P}(\mathbf{X}) := \mathrm{col}(\mathbf{p}(\mathbf{X})) = \mathbf{0}\,, \quad \mathbf{X} = (x_{jk}), \quad j,k = 1,2\,,$$

i.e. we solve

$$\mathbf{P}(\mathbf{X}) + \mathbf{P}'(\mathbf{X})\mathbf{S} = \mathbf{0}\,, \quad \mathrm{col}(\mathbf{X}) := \mathrm{col}(\mathbf{X}) + \mathbf{S}\,, \tag{24}$$

where the matrix $\mathbf{P}'$ is the corresponding Jacobi matrix. The Jacobi matrix **P**' can be found explicitly in a very simple way by using a technique described in [6], without employing partial derivatives.

In the following example, the computations were carried out with MATLAB.

Example We will treat a parameter dependent problem defined by

$$\mathbf{p}(\mathbf{X}(\lambda)) := \mathbf{A}_2\mathbf{X}^2\mathbf{B}_2 + \mathbf{A}_1\mathbf{X}\mathbf{B}_1 + \mathbf{C}(\lambda), \tag{25}$$

where

$$\mathbf{A}_2 := \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \qquad \mathbf{B}_2 := \begin{pmatrix} 5 & 10 \\ 4 & 8 \end{pmatrix}, \tag{26}$$

$$\mathbf{A}_1 := \begin{pmatrix} 9 & 11 \\ 10 & 12 \end{pmatrix}, \quad \mathbf{B}_1 := \begin{pmatrix} 13 & 15 \\ 14 & 16 \end{pmatrix}, \tag{27}$$

$$\mathbf{C}(\lambda) := -\begin{pmatrix} 288 & 345 \\ 324 & 394+\lambda \end{pmatrix}, \quad \lambda \in [-1, 1]. \tag{28}$$

Note, that $\mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_1\mathbf{B}_1 + \mathbf{C}(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix}$. If we denote the zeros by $\mathbf{X}(\lambda)$, we see that $\mathbf{X}(0) = \mathbf{I}$ is one of the zeros. The corresponding matrices $\mathbf{M}$, $\mathbf{N}$ from (16) and (17) for the zero $\mathbf{I}$ are

$$\mathbf{M} = \begin{pmatrix} 127 & 173 & 134 & 178 \\ 150 & 196 & 156 & 200 \\ 155 & 225 & 160 & 224 \\ 190 & 260 & 192 & 256 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} -305 \\ -350 \\ -379 \\ -446 \end{pmatrix}; \quad \mathbf{M}\mathrm{col}(\mathbf{I}) + \mathbf{N} = \mathbf{0} \text{ holds}.$$

In this case $\mathrm{rank}(\mathbf{M}) = 4$, i.e. in this case for $\lambda = 0$ the matrix $\mathbf{I}$ is the isolated zero.

However, there is another zero for $\lambda = 0$. For this zero the two matrices are

$$\mathbf{M} = \frac{1}{8}\begin{pmatrix} 931 & 1129 & 1004 & 1220 \\ 1030 & 1228 & 1112 & 1328 \\ 1070 & 1290 & 1144 & 1384 \\ 1180 & 1400 & 1264 & 1504 \end{pmatrix}, \quad \mathbf{N} = \frac{1}{8}\begin{pmatrix} -2151 \\ -2358 \\ -2454 \\ -2684 \end{pmatrix}, \quad \mathbf{M}\mathrm{col}(\mathbf{I}) + \mathbf{N} = \mathbf{0} \text{ holds, too.}$$

Here, $\mathrm{rank}(\mathbf{M}) = 3$, i.e. $\mathbf{I}$ is the zero of rank 3.

The general solution of $\mathbf{M}\mathrm{col}(\mathbf{X})+\mathbf{N}=\mathbf{0}$ has the form $\mathrm{col}(\mathbf{X})=\alpha\mathbf{x}_0+\mathbf{x}_1$ for all $\alpha\in\mathbb{R}$, where

$$\mathbf{x}_1=\frac{1}{11}\begin{pmatrix}-1\\12\\11\\0\end{pmatrix}\quad\text{and}\quad\mathbf{x}_0=\begin{pmatrix}0.52124669131568\\-0.52124669131568\\-0.47780946703938\\0.47780946703938\end{pmatrix}.$$

Acknowledgements

The research was supported by the German Science Foundation, DFG, under the contract number OP 33/19-1.

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