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YEAST GRAPHS AND FERMENTATION OF ALGEBRAIC LATTICES

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We introduce two novelties in the field of representation of algebraic lattices by congruences of algebras: First, when representing an algebraic lattice L as a closure system $\mathscr{L}$, we split the corresponding closure operator into two components so that the "closing work" is done by combinedly using a quasiordering, producing sets "closed below" and a dependency relation indicating how finite sets should be closed. A set J together with thus combined closure is called a "quadricle". Second, graphs with edges labelled by elements of a quadricle J are used as a device creating a congruence representation of the lattice L represented by J.

A wide class of representation creating graphs, called "yeast graphs" because of the resemblance of their construction with the process of fermentation, is shown to provide readily congruence lattice representations for all algebraic lattices, proving thus anew the well-known Grätzer – Schmidt Theorem (Section 1). By a sophisticated use of yeast graphs we were able to prove that all strongly representable lattices (i.e. the lattices L such that every complete sublattice $\mathscr{L}$ of a lattice Eq (A) of all equivalences on A, which is isomorphic to L, is a congruence lattice of some algebra on A) are completely distributive (Section 3). Yeast graphs yield also a new class of finite lattices representable as congruence lattices of finite algebras: the class $\mathfrak{F}$ of "finitely fermentable" lattices formed by those finite lattices L whose congruence lattices $\mathrm{Con}(L)$ have the same number of join irreducible elements as L (generally, it can never be greater) (Section 2). When dealing with representation problems, it is desirable to have more insight into the structure of equivalence lattices. We bring a sort of local characteristic which might be useful in this regard.

Exciting and persisting Birkhoff's problems of finite representation of lattices have lent the impetus to the present work.

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1. GRAPHS CREATING CONGRUENCE LATTICE REPRESENTATIONS

A *congruence lattice representation* of a given algebraic lattice L consists in pointing out an algebra A and a lattice isomorphism $\varphi: L \to \mathrm{Con}(A)$ of L onto the lattice of all congruences of A.

In this section, we shall describe an effective method of finding congruence lattice representations.

Let J be a set, let $P_f(J)$ denote the set of all finite subsets of J. Any part $D \subseteq J \times P_f(J)$ such that $\alpha D \{\alpha\}$ for all $\alpha \in J$ is a *dependency relation* on J. An element $\alpha \in J$ is *D-dependent* on a subset $X \subseteq J$ if there exists $Y \subseteq X$ such that $\alpha D Y$. A subset $X \subseteq J$ is *D-closed* if any $\alpha \in J$ which is D-dependent on X is in X. The set of all D-closed subsets of J forms an algebraic closure system $\mathscr{D}$ on J. For the sake of convenience, we shall use the same symbol both for a closure system and for the closure operator derived from it. This will enable us to write $\mathscr{D}(X)$ for the least D-closed subset of J containing X – its *D-closure*.

If $\leqslant$ is a quasiordering on J, then a subset $X \subseteq J$, containing with any $\alpha \in X$ all $\beta \in J$ with $\beta \leqslant \alpha$, will be called $\leqslant$-closed. The $\leqslant$-closed subsets of J form another closure system (operator) $\mathscr{B}$.

These two ways of introducing closures, by a quasiordering and by a dependency relation, can be combined into the following concept, serving as our basic tool for handling algebraic lattices.

1.1. Definition. A *quadricle* is a quadruple $J = (J, \leqslant, D, \mathscr{L})$, where J is a set, $\leqslant$ is a quasiordering on J, D is a dependency relation on J, and $\mathscr{L}$ is the closure system on J formed by the sets both $\leqslant$-closed and D-closed.

Clearly, to every algebraic lattice L, a quadricle $(J, \leqslant, D, \mathscr{L})$ can be found with $\mathscr{L}$ lattice-isomorphic to L.

For a set A, let $P_2(A)$ and $P_1(A)$ denote the set of all subsets of A with exactly two elements, called the *edges* in A, and the set of all subsets of A consisting of a single element, the *singletons* in A, respectively.

1.2. Definition. For a given quadricle $J = (J, \leqslant, D, \mathscr{L})$, the category $\mathfrak{G}(J)$ of J-graphs and J-maps is defined as follows:

(a) A J-graph is a triple $A = (A, r_A, \nu_A)$, where A is a set, $r_A \subseteq$ $\subseteq P_2(A)$ is a set of edges in A, $\nu_A : r_A \to J$ is a function called *valuation* on r_A.

(b) For a pair of J-graphs $A = (A, r_A, \nu_A)$, $B = (B, r_B, \nu_B)$, a mapping $f: A \to B$ is a J-map, if for every edge $\{a_1, a_2\} \in r_A$ either $f(a_1) = f(a_2)$ or $\{f(a_1), f(a_2)\} \in r_B$ and $\nu_B\{f(a_1), f(a_2)\} \leqslant \nu_1\{a_1, a_2\}$. A is *J-isomorphic* to B, if there exists a *J-isomorphism* $g: A \to B$, i.e., g is a bijective J-map, the inverse $g^{-1} : B \to A$ is also a J-map. A is a

J-subgraph of B if $A \subseteq B$ and the insertion i: $A \to B$: $a \mapsto a$ is a J-map.

(c) The *algebra associated* with a J-graph (A, r, ν) is the unary algebra (A, F) with F equal to the set of all J-maps of A into itself.

1.3. Convention. When needed, a set $s \subseteq r$ of edges in a J-graph (A, r, ν) will be considered as the symmetric relation $\{(a, b) \mid \{a, b\} \in s\}$, and vice versa, a symmetric relation S may occasionally be regarded as a set of edges $\{\{a, b\} \mid (a, b) \in S,\ a \neq b\}$. Accordingly, ν may be taken as a symmetric function on the set of ordered pairs corresponding to the edges in r.

1.4. Lemma. *Let* (A, r, ν) *be a J-graph and* (A, F) *its associated algebra. Let* $\mathscr{E}_A$ *denote the equivalence closure operator on* $A \times A$, *corresponding to the full equivalence lattice* $\mathrm{Eq}\,(A)$. *Then for every* $X \in \mathscr{L}$, $\mathscr{E}_A(\nu^{-1}(X)) \in \mathrm{Con}\,(A, F)$.

Proof. Let $X \in \mathscr{L}$, $\{a, b\} \in \nu^{-1}(X)$. If $f(a) \neq f(b)$ for $f \in F$, then $\nu(f(a), f(b)) \leqslant \nu(a, b)$. Since X is $\leqslant$-closed, $(f(a), f(b)) \in \nu^{-1}(X)$, which means, under 1.3, that $\nu^{-1}(X)$ is a stable relation with regard to F, and so is its equivalence closure $\mathscr{E}_A(\nu^{-1}(X))$.

1.5. Definition. For a J-graph $(A, r, \nu) \in \mathfrak{G}(J, \leqslant, D, \mathscr{L})$ with the associated algebra (A, F), the mapping

$$\varphi\colon\ \mathscr{L} \to \mathrm{Con}\,(A, F)\colon\ X \mapsto \mathscr{E}_A(\nu^{-1}(X)),$$

existent by 1.4, will be called the *mapping associated* with (A, r, ν). A J-graph (A, r, ν) is *representation creating*, if its associated mapping φ: $\mathscr{L} \to \mathrm{Con}\,(A, F)$ is a lattice isomorphism.

Let us recall that, in an undirected graph (A, r), a sequence $x_1, \ldots, x_n \in r$ of distinct edges is a *path* of length n, *r-connecting* a to b, if there is a sequence $a_0, a_1, \ldots, a_n \in A$ of distinct points with $a = a_0$, $b = a_n$, and $x_i = \{a_{i-1}, a_i\}$ for $i = 1, \ldots, n$. A sequence $x_1, \ldots, x_n \in r$ of distinct edges is a *cycle,* if $x_2, \ldots, x_n$ is a path r-connecting the points of x_1. We shall write $a \leftrightarrow b$ in r, if either $a = b$ of there exists a path r-connecting a to b. The relation $\leftrightarrow$ is an equivalence on A called *r-connectedness* (under 1.3, it is just $\mathscr{E}_A(\boldsymbol{r})$), its classes are *r-components*. A is *r-connected* if A is a single r-component, otherwise A is r-disconnected.

For a unary algebra (A, F), let $\mathscr{C}_A$ denote the closure operator corresponding to $\mathrm{Con}\,(A, F)$, and $\mathscr{C}_A(a, b)$ the least congruence on (A, F) containing (a, b). If F is a monoid, then $\mathscr{C}_A(a, b) = \mathscr{E}_A\{(f(a), f(b)) \mid f \in F\}$.

1.6. Theorem. *Let* $(A, \boldsymbol{r}, \nu) \in \mathfrak{G}(J, \leqslant, D, \mathscr{L})$ *have the associated algebra* (A, F). *The following conditions are sufficient for* $(A, \boldsymbol{r}, \nu)$ *to be representation creating:*

RC 1: *Whenever* $\nu(x)\, D\, X$ *for some* $x \in \boldsymbol{r}$, $X \subseteq J$, *then there exists a path* $x_1, \ldots, x_n \in \boldsymbol{r}$ *connecting the points of* x, *with* $\{\nu(x_1), \ldots, \nu(x_n)\} = X$.

RC 2: *If* $x_1, \ldots, x_n \in \boldsymbol{r}$ *is a path connecting the points of* $x \in \boldsymbol{r}$, *then* $\nu(x) \in \mathscr{L}\{\nu(x_1), \ldots, \nu(x_n)\}$.

RC 3: *The valuation* ν *maps* $\boldsymbol{r}$ *onto* J.

RC 4: *If* $\nu(x) \geqslant \nu(y)$ *for some* $x, y \in \boldsymbol{r}$, *then there exists a* J*-map* $f \in F$ *which maps* x *onto* y.

RC 5: *For every* $a, b \in A$, $a \leftrightarrow b$ *in* $\boldsymbol{r} \cap \mathscr{C}_A(a, b)$.

Proof. We show first that $\nu(\boldsymbol{r} \cap E) \in \mathscr{L}$ for $E \in \mathrm{Con}\,(A, F)$. If $\beta \leqslant \alpha = \nu(x)$ for some $x \in \boldsymbol{r} \cap E$, then by RC 3 there exists $y \in \boldsymbol{r}$ with $\nu(y) = \beta$. By RC 4, some $f \in F$ maps x onto y, hence $y \in E$ and $\beta \in \nu(\boldsymbol{r} \cap E)$, which proves $\nu(\boldsymbol{r} \cap E)$ is $\leqslant$-closed. If $\alpha\, D\, X$ for some $X \subseteq \nu(\boldsymbol{r} \cap E)$, then by RC 3, $\alpha = \nu(x)$ for some $x \in \boldsymbol{r}$, by RC 1 there exists a path $x_1, \ldots, x_n \in \boldsymbol{r}$ connecting the points of x, with $\{\nu(x_1), \ldots, \nu(x_n)\} = X$. There are $y_i \in \boldsymbol{r} \cap E$ with $\nu(x_i) = \nu(y_i)$, thus mapped onto x_i by some $f_i \in F$, $i = 1, \ldots, n$ existent by RC 4. Since $E \in \mathrm{Con}\,(A, F)$, it is $x_1, \ldots, x_n \in E$, hence $x \in E$, and $\alpha = \nu(x) \in \nu(\boldsymbol{r} \cap E)$ proves X is D-closed.

Next we take the mapping

$$\psi\colon \mathrm{Con}\,(A, F) \to \mathscr{L} : E \mapsto \nu(r \cap E)$$

and prove that it is inverse to

$$\varphi\colon \mathscr{L} \to \mathrm{Con}\,(A, F)\colon X \mapsto \mathscr{E}_A(\nu^{-1}(X))$$

associated with (A, r, ν).

For any $X \in \mathscr{L}$, $\psi\varphi(X) = \nu(r \cap \mathscr{E}_A(\nu^{-1}(X)) \supseteq X$. To show the converse inclusion, for $x \in r \cap \mathscr{E}_A(\nu^{-1}(X))$ there is a path $x_1, \ldots, x_n \in \in \nu^{-1}(X)$ connecting the points of x, hence by RC 2, $\nu(x) \in \in \mathscr{L}\{\nu(x_1), \ldots, \nu(x_n)\} \subseteq \mathscr{L}(X) = X$.

For any $E \in \mathrm{Con}\,(A, F)$, $\varphi\psi(E) = \mathscr{E}_A \nu^{-1}\nu(r \cap E) \supseteq \mathscr{E}_A(r \cap E)$. If $(a, b) \in \mathscr{E}_A \nu^{-1}\nu(r \cap E)$, then there is a path $y_1, \ldots, y_n \in r$ connecting a to b such that $\nu(y_i) = \nu(x_i)$ for some $x_i \in r \cap E$, $i = 1, \ldots, n$. By RC 4, each x_i is mapped onto y_i by some $g_i \in F$, therefore $y_1, \ldots, y_n \in E$ and thus $(a, b) \in \mathscr{E}_A(r \cap E)$, which proves $\varphi\psi(E) = \mathscr{E}_A(r \cap E)$. Clearly $\mathscr{E}_A(r \cap E) \subseteq E$. On the other hand, for $(a, b) \in E$, by RC 5 we have $a \leftrightarrow b$ in $r \cap \mathscr{C}_A(a, b) \subseteq r \cap E$, hence $(a, b) \in \mathscr{E}_A(r \cap E)$.

Since both φ and ψ are isotone with regard to the orderings of $\mathscr{L}$ and $\mathrm{Con}\,(A, F)$ by the relation of inclusion, they are both lattice isomorphisms.

For a big class of quadricles J, representation creating J-graphs can be constructed inductively from very simple J-graphs called "cells".

1.7. Definition. An *α-cell* is a J-graph $(C_x, r_x, \nu_x) \in \mathfrak{G}(J, \leqslant, D, \mathscr{L})$ in which for a distinguished edge $x \in r_x$ called the *base* of C_x' valuated by $\alpha = \nu_x(x)$ and written as a subscript, the set $\mathscr{P}_x$ of all paths r_x-connecting the points of the base x satisfies the next three conditions:

C 1: There are only the paths $x_1, \ldots, x_p$ with $\alpha\, D\, \{\nu_x(x_1), \ldots, \nu_x(x_p)\}$ in $\mathscr{P}_x$ and for every $Y \subseteq J$ such that $\alpha\, D\, Y$ there is a path $y_1, \ldots, y_q$ in $\mathscr{P}_x$ with $\{\nu_x(y_1), \ldots, \nu_x(y_q)\} = Y$.

C 2: Every edge $y \in r_x$, $y \neq x$ is in exactly one path from $\mathscr{P}_x$.

C 3: For every path $x_1, \ldots, x_p$ in $\mathscr{P}_x$ the following "cutting property" holds: For any i, j with $1 \leqslant i \leqslant j \leqslant p$ either

$$\{\nu_x(x_1), \ldots, \nu_x(x_{i-1})\} \cup \{\nu_x(x_{j+1}), \ldots, \nu_x(x_n)\} =$$
$$= \{\nu_x(x_1), \ldots, \nu_x(x_p)\},$$

or

$$\{\nu_x(x_i), \ldots, \nu_x(x_j)\} = \{\nu_x(x_1), \ldots, \nu_x(x_p)\}.$$

A class $\mathfrak{C}$ of α-cells is *suited* for J, if:

S 1: $\mathfrak{C}$ is closed under J-isomorphisms.

S 2: For every $\alpha \in J$ there is an α-cell in $\mathfrak{C}$.

S 3: If $(C_x, r_x, \nu_x), (C_y, r_y, \nu_y) \in \mathfrak{C}$ and $\nu_x(x) \geqslant \nu_y(y)$ then every mapping $f\colon x \to y$ can be extended to a J-map $g\colon C_x \to C_y$.

1.8. Construction. Let $\mathfrak{C}$ be a class of α-cells suited for a quadricle J. Construct a sequence $\mathfrak{A} = \{(A_n, r_y, \nu_n) \mid n \geqslant 0\}$ of J-graphs called "growths" inductively as follows:

I. Set $A_0 = J \cup \{\omega\}$ for some $\omega \notin J$, $r_0 = \{\{\alpha, \omega\} \mid \alpha \in J\}$, $\nu_0(\alpha, \omega) = \alpha$ for $\alpha \in J$. Call A_0 the *germ* of $\mathfrak{A}$.

II. Assume that (A_n, r_n, ν_n) has already been constructed. Let us call $s_n = r_n \setminus r_{n-1}$, $s_0 = r_0$ the *surface* and the edges contained in s_n the *surface edges* of A_n. Properties S1, S2 of 1.7 enables us to choose a family $\{(C_x^{n+1}, r_x^{n+1}, \nu_x^{n+1}) \in \mathfrak{C} \mid x \in s_n\}$ with $C_x^{n+1} \cap A_n = x$, $\nu_x^{n+1}(x) = \nu_n(x)$, $C_x^{n+1} \cap C_y^{n+1} = x \cap y$ for any $x, y \in s_n$, $x \neq y$ called the *$(n+1)$-th generation* of cells for $\mathfrak{A}$. Set

$$A_{n+1} = A_n \cup \bigcup_{x \in s_n} C_x^{n+1}, \quad r_{n+1} = r_n \cup \bigcup_{x \in s_n} r_x^{n+1},$$

$$\nu_{n+1}(y) = \begin{cases} \nu_n(y) & \text{if } y \in r_n \\ \nu_x^{n+1}(y) & \text{if } y \in r_x^{n+1}, \quad x \in s_n \end{cases}$$

and call A_{n+1} the *$(n+1)$-th growth* of $\mathfrak{A}$.

Note that $A_n \subseteq A_{n+1}$, $r_n \subseteq r_{n+1}$, and ν_{n+1} is a correctly defined extension of ν_n, since two distinct cells of $(n+1)$-th generation have no edges in common and for $y \in r_n \cap r_x^{n+1}$ it is $y = x$ therefore $\nu_{n+1}(x) = \nu_x^{n+1}(x) = \nu_n(x)$. Hence each growth A_n is a J-subgraph of the next growth A_{n+1}.

1.9. Lemma. *Let* $\mathfrak{A} = \{A_0, A_1, \ldots, A_n, \ldots\}$ *be a sequence of growths obtained by* 1.8. *Then*

(1) *every* A_n, $n \geqslant 0$, *satisfies condition* RC 5 *of* 1.6.

(2) *for any surface edge* $y \in s_n$ *there exists a J-map* $f\colon A_n \to A_n$ *with* $f(A_n) = f(y) = y$.

Proof. Let $\mathscr{C}_n$ denote the congruence closure on A_n corresponding to the algebra associated with A_n, and proceed by induction on n.

I. For A_0, there is for every $\{\alpha, \omega\} \in s_0$ a J-map $f_\alpha\colon A_0 \to A_0$ with $f_\alpha(\alpha) = \alpha$ and $f_\alpha(\beta) = \omega$ if $\beta \neq \alpha$, projecting A_0 onto $\{\alpha, \omega\}$. Let $\alpha, \beta \in A_0$, $\alpha \neq \beta$. If $\{\alpha, \beta\} \in r_0$, then evidently $\alpha \longleftrightarrow \beta$ in $r_0 \cap \cap \mathscr{C}_0(\alpha, \beta)$. If $\alpha \neq \omega \neq \beta$, then $\{\alpha, \omega\}, \{\omega, \beta\}$ is a path r_0-connecting α to β and $\{\alpha, \omega\} = \{f_\alpha(\alpha), f_\alpha(\beta)\}$, $\{\omega, \beta\} = \{f_\beta(\alpha), f_\beta(\beta)\}$ hence again $\alpha \longleftrightarrow \beta$ in $r_0 \cap \mathscr{C}_0(\alpha, \beta)$.

II. Assume the assertion true for A_n. For $a, b \in A_{n+1}$, $a \neq b$ consider two cases:

(1) Assume first that $a, b \in \bigcup_{k=1}^{p} x_k$ for some path $x_1, \ldots, x_p \in \in r_x^{n+1}$ connecting the points of the base of some $C_x^{n+1} \subseteq A_{n+1}$, say, $x_k = \{c_{k-1}, c_k\}$ for a sequence $c_0, c_1, \ldots, c_p \in C_x^{n+1}$ of distinct points and $a = c_{i-1}$, $b = c_j$ for some $i \leqslant j$. Then $\{c_0, c_p\} = x$, the path $x_1, \ldots, x_{i-1}$ connects c_0 to a, $x_i, \ldots, x_j$ connects a to b, and $x_{j+1}, \ldots, x_p$ connects b to c_p. If the first alternative in C 3 of 1.7 occurs, we can find to every edge $x_k \in \{x_i, \ldots, x_j\}$ an edge $y_k \in \in \{x_1, \ldots, x_p\} \setminus \{x_i, \ldots, x_j\}$ with $\nu_{n+1}(y_k) = \nu_{n+1}(x_k)$. By C 2 of 1.7, the removal of x_k and y_k from r_{n+1} makes A_{n+1} $(r_{n+1} \setminus \{x_k, y_k\})$-disconnected with two components, one containing c_{k-1} and the other

c_k. Hence the mapping $f_k \colon A_{n+1} \to A_{n+1}$ collapsing the whole component of c_{k-1} onto $\{c_{k-1}\}$ and the rest onto $\{c_k\}$ is a J-map, since the only edges in r_{n+1} which are not collapsed by f_k to a singleton are x_k and y_k with $\nu_{n+1}(x_k) = \nu_{n+1}(y_k)$ and $f_k(y_k) = f_k(x_k) = x_k$. Since $x_i, \ldots, x_{k-1}$ connects a to c_{k-1} and $x_{k+1}, \ldots, x_j$ connects c_k to b in $r_{n+1} \setminus \{x_k, y_k\}$, it is $f_k(a) = c_{k-1}$ and $f_k(b) = c_k$ for all $k = i, \ldots, j$ which means that $x_i, \ldots, x_j$ connects a to b in $r_{n+1} \cap$ $\cap\, \mathscr{C}_{n+1}(a, b)$.

If the second alternative in C 3 of 1.7 takes place, we can show by a similar argument that for every $k \in \{1, \ldots, i-1\} \cup \{j+1, \ldots, p\}$ there exists a J-map $f_k \colon A_{n+1} \to A_{n+1}$ projecting A_{n+1} and $\{a, b\}$ onto x_k, hence $c_0 \leftrightarrow a$ in $r_{n+1} \cap \mathscr{C}_{n+1}(a, b)$ and $b \leftrightarrow c_p$ in $r_{n+1} \cap$ $\cap\, \mathscr{C}_{n+1}(a, b)$, whence $(c_0, c_p) \in \mathscr{C}(a, b)$ and we are done. Note that the second assertion of the lemma has been completely settled.

(2) Assume further $a \in \bigcup_{k=1}^{p} x_k$ as in (1) and $b \notin \bigcup_{k=1}^{p} x_k$. By C 3 of 1.7 again, either $x_1, \ldots, x_{i-1}$ connecting c_0 to a or the complementary path $x_i, \ldots, x_p$ connecting a to c_p admits J-maps projecting A_{n+1} and $\{a, b\}$ onto its edges, hence $a \leftrightarrow c$ in $r_{n+1} \cap \mathscr{C}_{n+1}(a, b)$ for some $c \in x \subseteq A_n$. For the same reason, $b \leftrightarrow d$ in $r_n \cap \mathscr{C}_n(a, b)$ for some $d \in A_n$, thus $(c, d) \in \mathscr{C}_{n+1}(a, b)$. By induction hypothesis, $c \leftrightarrow d$ in $r_n \cap \mathscr{C}_n(c, d)$. Since every J-map $f \colon A_n \to A_n$ can be due to S 3 of 1.7. extended to a J-map $g \colon A_{n+1} \to A_{n+1}$, it follows that $\mathscr{C}_n(c, d) \subseteq \mathscr{C}_{n+1}(c, d) \subseteq \mathscr{C}_{n+1}(a, b)$ and we are done.

In general, growths need not enjoy all the conditions listed in 1.6, however, this is easily remedied by taking their limit, which we call a "yeast graph" in order to evoke the process of "growing cells" by successive "generations" attached to the "surfaces" of preceding "growths", starting from a "germ".

1.10. Definition. Let $\mathfrak{C}$ be a class of α-cells suited for a quadricle J, let $\{(A_n, r_n, \nu_n) \mid n \geqslant 0\}$ be a sequence of J-graphs constructed by 1.8. The J-graph (A, r, ν), where $A = \bigcup_{n=1}^{\infty} A_n$, $r = \bigcup_{n=1}^{\infty} r_n$ and $\nu \colon r \to J$ is the unique common extension of all ν_n, $n \geqslant 0$, will be called a *yeast J-graph* over $\mathfrak{C}$.

1.11. Theorem. *Let* (A, r, ν) *be a yeast J-graph over a class* $\mathfrak{C}$ *of* α*-cells suited for a quadricle* $(J, \leqslant, D, \mathscr{L})$. *Then* (A, r, ν) *is a representation creating J-graph.*

Proof. We will show that (A, r, ν) verifies conditions RC 1-RC 5 of 1.6. Let (A, F) be the algebra associated with (A, r, ν) and let $\mathscr{C}_A$ denote the congruence closure operator corresponding to $\mathrm{Con}\,(A, F)$.

(1) If $\nu(x)\, D\, X$ for some $x \in r$, then $x \in s_n$ for some $n \geqslant 0$ and x is the base of a cell C_x^{n+1} contained in A as a J-subgraph. By C 1 of 1.7, there is a path $x_1, \ldots, x_p \in r_x^{n+1} \subseteq r$ connecting the points of x with $\{\nu(x_1), \ldots, \nu(x_p)\} = X$, hence A verifies RC 1.

(2) Let $x_1, \ldots, x_p \in r$ be a path connecting the points of $x \in r$ and proceed by the following inductive argument on the least n such that $\{x, x_1, \ldots, x_p\} \subseteq r_n$: A_0 verifies RC 2, since there are no cycles in r_0. Assume further RC 2 for A_n and let $\{x, x_1, \ldots, x_p\} \subseteq r_{n+1}$. If $x \in s_{n+1}$, then x is contained in a path $y_1, \ldots, y_q \in s_{n+1}$, unique by C 2 of 1.7, connecting the points of some edge $y \in s_n$, hence $\{x, x_1, \ldots, x_p\} \supseteq \{y_1, \ldots, y_q\}$. By C 3 of 1.7, for each y_i there exists y_j in $y_1, \ldots, y_q$ with $j \neq i$ and $\nu(y_j) = \nu(y_i)$, hence $\nu(x) \in$ $\in \{\nu(x_1), \ldots, \nu(x_n)\}$.

If $x \in r_n$, replace each maximal segment $x_{i_k}, \ldots, x_{j_k} \in r_{n+1}$ of $x_1, \ldots, x_n$ by the edge $y_k \in s_n$ connected by it, getting thus a path $z_1, \ldots, z_r \in r_n$ connecting points of x. By hypothesis, $\nu(x) \in$ $\in \mathscr{L}\{\nu(z_1), \ldots, \nu(z_r)\}$. By C 1 of 1.7, $\nu(y_k)\, D\, \{\nu(x_{i_k}), \ldots, \nu(x_{j_k})\}$. hence $\{\nu(z_1), \ldots, \nu(z_r)\} \subseteq \mathscr{L}\{\nu(x_1), \ldots, \nu(x_p)\}$.

(3) RC 3 is satisfied even by $A_0 \subseteq A$.

(4) Let $\nu(x) \geqslant \nu(y)$ for $x \in s_n$, $y \in r_m$. By 1.9, there exists an J-map $f\colon A_n \to A_n$ with $f(A_n) = f(x) = x$. For an arbitrary bijection g of x into y, the composite $gf\colon A_n \to A_m$ is a J-map again and it can

be extended, using S 3 of 1.7 repeatedly in a simple induction scheme, to some $h \in F$ with $h(x) = y$.

(5) Let $a, b \in A$. It must be $a, b \in A_n$ for some $n \geqslant 0$. By 1.9, $a \leftrightarrow b$ in $r_n \cap \mathscr{C}_n(a, b)$. By S 3 of 1.7, every J-map $f\colon A_n \to A_n$ can be extended to some $g \in F$, hence $\mathscr{C}_n(a, b) \subseteq \mathscr{C}(a, b)$.

Existence of a system $\mathfrak{C}$ of α-cells suited for $(J, \leqslant, D, \mathscr{L})$ imposes to J a certain relationship between $\leqslant$ and D.

1.12. Definition. Let $(J, \leqslant, D, \mathscr{L})$ be a quadricle. Let us write $Y \prec X$ for $X, Y \in P_f(J)$, if for every $\beta \in Y$ there exists $\alpha \in X$ with $\beta \leqslant \alpha$. Call J a *commutable* quadricle, if for any $\alpha, \beta \in J$ and $X \in P_f(J)$ such that $\beta \leqslant \alpha$ and $\alpha\, D\, X$, there exists $Y \in P_f(J)$ such that $Y \prec X$ and $\beta\, D\, Y$. Put otherwise, the next diagram can be always completed:

$$\begin{array}{ccc} \alpha & \longrightarrow & X \in P_f(J) \\ \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}} & & \curlyvee \\ \beta & \longrightarrow & \quad \in P_f(J) \end{array}$$

in a commutable quadricle J.

1.13. Lemma. *If there exists a class* $\mathfrak{C}$ *of* α*-cells suited for* $(J, \leqslant, D, \mathscr{L})$, *then* J *is commutable.*

Proof. Let $\beta \leqslant \alpha$, $\alpha\, D\, X$. By S 2 of 1.7, there are $C_x, C_y \in \mathfrak{C}$ with $\nu_x(x) = \alpha$, $\nu_y(y) = \beta$. By C 1 we have $X = \{\nu_x(x_1), \ldots, \nu_x(x_p)\}$ for some path $x_1, \ldots, x_p \in r_x$ connecting the points of x. By S 3, there exists a J-map $g\colon C_x \to C_y$ with $g(x) = y$. If $x_{i_1}, \ldots, x_{i_q}$ is a subsequence of $x_1, \ldots, x_p$ formed by all those edges which are not collapsed to singleton by g, then $g(x_{i_1}), \ldots, g(x_{i_q})$ is a path r_y-connecting the points of y, hence by C 1 again,

$$\beta = \nu_y(y)\, D\, \{\nu_y(g(x_{i_1})), \ldots, \nu_y(g(x_{i_q}))\} \prec X.$$

Clearly, any quadricle $(J, =, D, \mathscr{D})$ in which the quasiordering is the equality on J, is commutable.

Now, it is very easy to construct a class of α-cells suited for J in this sort of "degenerate" case.

1.14. Construction. Let J be a quadricle with a dependency relation D. Call an *αD-list* in J any sequence $\sigma = (\alpha_1, \ldots, \alpha_s)$ of distinct elements of J and of length $l(\sigma) = s \geqslant 2$, such that $\alpha D \{\alpha_1, \ldots, \alpha_s\}$. Let $\langle \alpha D \rangle$ denote the set of all αD-lists in J. For $n > 1$ construct a J-graph $(C_x^{(n)}, r_x, \nu_x)$ with a distinguished edge $x = \{a, b\} \in r_x$, as follows. Assign to each $\sigma = (\alpha_1, \ldots, \alpha_s) \in \langle \alpha D \rangle$ a J-graph $(P_\sigma^{(n)}, r_\sigma, \nu_\sigma)$, called an n-fold σ-path, with

$$P_\sigma^{(n)} = \{c_0^\sigma, c_1^\sigma, \ldots, c_{n \cdot l(\sigma)}^\sigma\}, \quad r_\sigma = \{x_i \mid x_i = \{c_{i-1}^\sigma, c_i^\sigma\},$$
$$i = 1, \ldots, n \cdot l(\sigma)\},$$

$$\nu_\sigma(x_i) = \alpha_k \quad \text{if} \quad i \equiv k \pmod{l(\sigma)}, \; i = 1, \ldots, n \cdot l(\sigma),$$
$$k = 1, \ldots, l(\sigma),$$

in such a way, that $c_0^\sigma = a$, $c_{n \cdot l(\sigma)}^\sigma = b$ for all $\sigma \in \langle \alpha D \rangle$ and $P_\sigma^{(n)} \cap P_\tau^{(n)} = \{a, b\}$ for any $\sigma, \tau \in \langle \alpha D \rangle$, $\sigma \neq \tau$. Set $C_x^{(n)} = \bigcup_{\sigma \in \langle \alpha D \rangle} P_\sigma^{(n)}$, $r_x = \{x\} \cup \bigcup_{\sigma \in \langle \alpha D \rangle} r_\sigma$, $\nu_x(x) = \alpha$, $\nu_x(y) = \nu_\sigma(y)$ if $y \in r_\sigma$. For $\alpha \in J$, let $\mathfrak{C}_J^\alpha(n)$ denote the class of all J-graphs J-isomorphic to $(C_x^{(n)}, r_x, \nu_x)$. Finally, set $\mathfrak{C}_J(n) = \bigcup_{\alpha \in J} \mathfrak{C}_J^\alpha(n)$.

1.15. Theorem. *Let D be an arbitrary dependency relation on J, let $\mathfrak{D}$ denote the corresponding algebraic lattice of D-closed sets. Then $\mathfrak{D}$ has a congruence lattice representation. Moreover, for every $n \geqslant 2$ the class $\mathfrak{C}_J(n)$ of* 1.14 *is a class of α-cells suited for $(J, =, D, \mathfrak{D})$, therefore $\mathfrak{D}$ admits congruence lattice representations created by yeast graphs over $\mathfrak{C}_J(n)$.*

Proof. It is easily seen that $(C_x^{(n)}, r_x, \nu_x)$ of 1.14 satisfies C 1, C 2, C 3 of 1.7. It is C 3 that requires at least two-fold σ-paths. For $\mathfrak{C}_J(n)$, S 1 and S 2 of 1.7 are obvious. To show S 3, note that $C_x^{(n)}$ has a J-involution interchanging the points of the base x and carrying each

$(\alpha_1, \ldots, \alpha_s)$-path in $\mathscr{P}_x$ to $(\alpha_s, \alpha_{s-1}, \ldots, \alpha_1)$-path corresponding to the reversed αD-list. Hence $\mathfrak{C}_J(n)$ is suited for J and 1.11 applies.

2. FINITELY FERMENTABLE LATTICES

Construction 1.8 may happen to "terminate" on its n-th step, in the sense that it may be possible to choose $(n+1)$-th generation on α-cells in $\mathfrak{C}$ consisting only on their bases. Then, repeating the same choice of "terminal" α-cells for all subsequent steps, we shall have $A_n = A_{n+k}$ for all $k \geqslant 0$, hence the n-th growth A_n will be the resulting yeast graph. If this happens for a finite quadricle $(J, \leqslant, D, \mathscr{L})$, then A_n is a finite yeast J-graph creating a representation of $\mathscr{L}$ by the congruence lattice of a *finite* algebra – the one associated with A_n.

In this section we shall specify the class $\mathfrak{F}$ of finite lattices which can be represented by quadricles admitting finite yeast graphs. The members of $\mathfrak{F}$ will be called *finitely fermentable lattices.*

The following property of J proves crucial:

2.1. Definition. A quadricle $(J, \leqslant, D, \mathscr{L})$ will be called *cyclic* if there exists a subset $Z \subseteq J$, such that for any $\alpha \in Z$ there exists $\beta \in Z$, $\beta \neq \alpha$, such that $\alpha\, D\, (\{\beta\} \cup X)$ for some $X \subseteq J$.

2.2. Lemma. *If a quadricle* $(J, \leqslant, D, \mathscr{L})$ *is cyclic, then it does not admit a finite yeast graph.*

Proof. Let Z have the property stated in 2.1. In any class $\mathfrak{C}$ of cells suited for J, the α-cells in $\mathfrak{C}$ with $\alpha \in Z$ must contain an edge with a value $\neq \alpha$. A simple induction shows that every generation of cells used in 1.8 must contain an α-cell with $\alpha \in Z$.

The above lemma suggests an idea to carry over to the quasiordering $\leqslant$ as much as possible of the "closing work" done by the dependency relation D, in a quadricle $(J, \leqslant, D, \mathscr{L})$ designed for a finite congruence lattice representation of a given lattice L. This can be done in the following standard way: First represent L as a closure system $\mathscr{L}$ on the set J of all join-irreducible elements of L, by the assignment $\alpha \mapsto [\alpha] =$
$= \{\beta \mid \beta \leqslant \alpha \text{ and } \beta \in J\}$ for $\alpha \in L$. The $\mathscr{L}$-closure of a subset $X \subseteq J$ is

then $\mathscr{L}(X) = [\sup X]$. Next, take the greatest quasiordering $\leqslant$ on J such that all $\leqslant$-closed sets belong to $\mathscr{L}$, i.e., $\alpha \leqslant \beta$ if and only if $\mathscr{L}\{\alpha\} \subseteq \mathscr{L}\{\beta\}$. This quasiordering $\leqslant$ turns out to be the partial ordering induced on J by that of L. Finally, introduce the dependency relation D on J with $\alpha D \{\alpha_1, \ldots, \alpha_n\}$ if and only if $\alpha \leqslant \alpha_1 \vee \ldots \vee \alpha_n$ and $\alpha \nleqslant \alpha_1 \vee \ldots \vee \alpha_{i-1} \vee \hat{\alpha}_i \vee \alpha_{i+1} \vee \ldots \vee \alpha_n$ for $i = 1, \ldots, n$, where $\hat{\alpha}_i$ denotes the unique element in L covered by α_i. Put otherwise, $\alpha D X$ if and only if X is minimal both relative to $\subseteq$ and to $\prec$ (induced on the subsets of J in the sense of 1.12) with $\alpha \in \mathscr{L}(X)$. Let us call this particular D the *congruence dependency relation* on J.

2.3. Lemma. *It is* $\alpha \in \mathscr{L}(X)$ *if and only if* $\alpha D Y$ *for some* $Y \subseteq \bigcup_{\beta \in X} \mathscr{L}\{\beta\}$. *Consequently,* $X \in \mathscr{L}$ *if and only if* X *is both* $\leqslant$*-closed and* D*-closed.*

Proof. If $\alpha D Y$ for some $Y \subseteq \bigcup_{\beta \in X} \mathscr{L}\{\beta\}$, then clearly $\alpha \in \mathscr{L}(X)$. Conversely, for $\alpha \in \mathscr{L}(X)$ there exists both $\subseteq$ and $\prec$-minimal $Y \subseteq J$ with $\alpha \in \mathscr{L}(Y)$ and $Y \prec X$, i.e., $\alpha D Y$ and $Y \subseteq \bigcup_{\beta \in X} \mathscr{L}\{\beta\}$. Every $X \in \mathscr{L}$ is both $\leqslant$ and D-closed. On the other hand, let X be both $\leqslant$ and D-closed. If $\alpha \in \mathscr{L}(X)$, then $\alpha D Y$ for some $Y \subseteq \bigcup_{\beta \in X} \mathscr{L}\{\beta\}$, but $\leqslant$-closedness of X means $\bigcup_{\beta \in X} \mathscr{L}\{\beta\} \subseteq X$, hence $\alpha D Y$ for some $Y \subseteq X$. The D-closedness of X implies $\alpha \in X$, hence $\mathscr{L}(X) \subseteq X$ and we are done.

2.4. Definition. Let L be a finite lattice, let J denote the set of all join irreducible elements of L. The quadricle $(J, \leqslant, D, \mathscr{L})$, where $\leqslant$ is the partial ordering induced on J by that of L, D is the congruence dependency relation on J and $\mathscr{L} \cong L$ by 2.3, will be called the *standard quadricle representing* L.

2.5. Theorem. *A finite lattice* L *is finitely fermentable if and only if there exists an acyclic and commutable quadricle* J_1 *representing* L. *In this case the standard quadricle* J *representing* L *admits a finite yeast* J*-graph.*

Proof. Assume first L finitely fermentable and let $(J_1, \leqslant_1, D_1, \mathscr{L}_1)$ be a quadricle admitting a finite yeast graph, with $\mathscr{L}_1 \cong L$. By 1.13, J_1 is commutable, by 2.2 it is acyclic. Define the "powers" of D_1 as follows: Let $D_1^{(1)} = D_1$. If $D_1^{(n)}$ is already defined, define $D_1^{(n+1)}$ by: $\alpha D_1^{(n+1)} Z$ if and only if there exist $X \subseteq J_1$ and a family $\{Y_\beta \subseteq J_1 \mid \beta \in X\}$ such that $\alpha D^{(n)} X$, $Z = \bigcup_{\beta \in X} Y_\beta$, and $\beta D_1^{(1)} Y_\beta$ for every $\beta \in X$. A simple induction shows every $D_1^{(n)}$, $n \geqslant 0$, commutable and acyclic. Since $D_1^{(n)} \subseteq D_1^{(n+1)}$ for all $n \geqslant 1$ and $J_1 \times P_f(J_1)$ is finite, there exists $m \geqslant 1$ and $D_1^* \subseteq J_1 \times P_f(J_1)$ such that $D_1^* = D_1^{(m+k)}$ for all $k \geqslant 0$. Since a set $X \subseteq J_1$ is D_1-closed if and only if it is D_1^*-closed, we can replace D_1 by D_1^*, getting thus a new quadricle $(J_1, \leqslant_1, D_1^*, \mathscr{L}_1)$ representing L, which is commutable, acyclic and enjoys the following property:

If $X_1, \ldots, X_m \in \mathscr{L}_1$, then $\alpha \in \mathscr{L}_1\left(\bigcup_{i=1}^m X_i\right)$ if and only if $\alpha D_1^* Y$ for some $Y \subseteq \bigcup_{i=1}^m X_i$.

Indeed, $\alpha D_1^* Y$ for $Y \subseteq \bigcup_{i=1}^m X_i$ means that α belongs to the D_1-closure of $\bigcup_{i=1}^m X_i$, hence $\alpha \in \mathscr{L}_1\left(\bigcup_{i=1}^m X_i\right)$. The converse implication will be proved, if we show the set $\{\alpha \mid \alpha D_1^* Y$ for some $Y \subseteq \bigcup_{i=1}^m X_i\}$ both $\leqslant$ and D_1-closed: If $\beta \leqslant \alpha$ and $\alpha D_1^* Y$ for $Y \subseteq \bigcup_{i=1}^m X_i$ then, by commutability $\beta D_1^* Z$ for some $Z \prec Y$, i.e., $Z \subseteq \bigcup_{i=1}^m X_i$. If $\beta D_1 \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_j D_1^* Y_j$ and $Y_j \subseteq \bigcup_{i=1}^m X_i$ for $j = 1, \ldots, n$, then $\beta D_1^* \left(\bigcup_{j=1}^n Y_j\right)$ and $\bigcup_{j=1}^n Y_j \subseteq \bigcup_{i=1}^m X_i$.

Assume now that the standard quadricle $(J, \leqslant, D, \mathscr{L})$ representing L is cyclic. We shall show that $(J_1, \leqslant_1, D_1^*, \mathscr{L}_1)$ is then also cyclic. The closure system $\mathscr{L}_1$ can be expressed in the form of a family $\mathscr{L}_1 = \{X_\alpha \mid \alpha \in L\}$ of subsets of J_1 indexed by elements of L in such a way that $X_\alpha \cap X_\beta = X_\gamma$ if and only if $\gamma = \alpha \wedge \beta$. Let a subset $Z \subseteq J$ have the property that for every $\alpha \in Z$ there exists $\beta \in Z$ such that $\alpha \nleqslant \beta$ and $\alpha D \{\beta\} \cup \{\gamma_1, \ldots, \gamma_n\}$ for some $\gamma_1, \ldots, \gamma_n \in J$. Form a subset $Z_1 = \bigcup_{\alpha \in Z} (X_\alpha \setminus X_{\hat{\alpha}})$ of the quadricle J_1 ($\hat{\alpha}$ denotes the element of L which is covered by α) and take some $a \in Z_1$, say $a \in X_\alpha \setminus X_{\hat{\alpha}}$ for some $\alpha \in Z$ such that $\alpha \leqslant \beta \vee \gamma_1 \vee \ldots \vee \gamma_n$ but $\alpha \nleqslant \hat{\beta} \vee \gamma_1 \vee \ldots \vee \gamma_n$. By the isomorphism of L onto $\mathscr{L}_1$ used for the indexation, we have $X_\alpha \subseteq \mathscr{L}_1(X_\beta \cup X_{\gamma_1} \cup \ldots \cup X_{\gamma_n})$ but $X_\alpha \nsubseteq \mathscr{L}_1(X_{\hat{\beta}} \cup X_{\gamma_1} \cup \ldots \cup X_{\gamma_n})$. By the property of D_1^* proved above, it is $a D_1^* Y$ for some $Y \subseteq \subseteq X_\beta \cap X_{\gamma_1} \cup \ldots \cup X_{\gamma_n}$. If it were $Y \cap (X_\beta \setminus X_{\hat{\beta}}) = \phi$ then it would be $X_\alpha = \mathscr{L}_1\{a\} \subseteq \mathscr{L}_1(X_{\hat{\beta}} \cup X_{\gamma_1} \cup \ldots \cup X_{\gamma_n})$, a contradiction. Hence for some $b \in X_\beta \setminus X_{\hat{\beta}}$ it is $a D_1^* \{b\} \cup (Y \setminus \{b\})$. Since $\alpha \neq \beta$, the sets $X_\alpha \setminus X_{\hat{\alpha}}$ and $X_\beta \setminus X_{\hat{\beta}}$ are disjoint, hence $a \neq b$, which proves J_1 cyclic. This contradiction shows that the standard quadricle representing a finitely fermentable lattice L is acyclic.

To prove the converse, assume that the standard quadricle $(J, \leqslant, D, \mathscr{L})$ representing a finite lattice L is acyclic. Assign to each $\alpha \in J$ its *multiplicity* $m(\alpha)$ inductively as follows:

(1) let minimal elements of $(J, \leqslant)$ have multiplicity 1;

(2) if for all $\beta < \alpha$, $m(\beta)$ is defined, set $m(\alpha) = \sum_{\beta < \alpha} m(\beta)$.

To get a system of α-cells suited for J, use a modification of 1.14 with αD-lists $\langle \alpha_1, \ldots, \alpha_p \rangle$ replaced by "multiple" αD-lists $\langle \beta_1, \ldots, \beta_s \rangle$ with $\beta_j = \alpha_k$ if and only if $\sum_{i=1}^{k-1} m(\alpha_i) < j \leqslant \sum_{i=1}^{k} m(\alpha_i)$ for $k = 1, \ldots, p$. Standard quadricles are always commutable, so the above modification of 1.14 easily provides for S 3 of 1.7. Acyclicity of J makes 1.8 to terminate after finite numbers of steps.

The dependency relation D in a standard quadricle J is closely related with congruences on the lattice L represented by J.

2.6. Definition. Let $(J, \leqslant, D, \mathscr{L})$ be a standard quadricle representing a finite lattice L. Call the transitive closure Q_D of the relation $\bigcup \{\{\alpha\} \times X \mid \alpha D X\}$ the *dependency* quasiordering on J. A subset $K \subseteq J$ is called *D-admissible* if for any $\alpha \in K$ and $X \subseteq J$, $\alpha D X$ implies $X \subseteq K$, i.e., if K is "closed upwards" with regard to Q_D. A quadruple $(K, \leqslant_K, D_K, \mathscr{L}_K)$, where K is a D-admissible subset of J, $\leqslant_K$ is the restriction of $\leqslant$ to K, $D_K = D \cap (K \times P_f(K))$, $\mathscr{L}_K(X) = K \cap \mathscr{L}(X)$ will be called the *restriction* of J to K.

2.7. Lemma. *If K is a D-admissible subset of a standard quadricle $(J, \leqslant, D, \mathscr{L})$, then the restriction $(K, \leqslant_K, D_K, \mathscr{L}_K)$ is a standard quadricle representing a lattice isomorphic to $\mathscr{L}_K$.*

Proof. We show first, that $(K, \leqslant_K, D_K, \mathscr{L}_K)$ is a quadricle, i.e., that $\mathscr{L}_K$ is formed by the subsets of K both $\leqslant_K$ and D_K-closed. Let $X \subseteq K$. If $\alpha \in \mathscr{L}_K(X) = K \cap \mathscr{L}(X)$, then $\alpha \in K$ and $\alpha \in \mathscr{L}(X) = \mathscr{L}\left(\bigcup_{\delta \in X} \mathscr{L}\{\delta\}\right)$, hence by 2.3, $\alpha \in K$ and $\alpha D Y$ for some $Y \subseteq \bigcup_{\delta \in X} \mathscr{L}\{\delta\}$. Since K is D-admissible, it is $Y \subseteq K$, therefore $\alpha \in \mathscr{L}_K(X)$ implies $\alpha D_K Y$ for some $Y \subseteq \bigcup_{\delta \in X} \mathscr{L}\{\delta\} \cap K = \bigcup_{\delta \in X} \mathscr{L}_K\{\delta\}$. Since $\beta \in \mathscr{L}_K\{\delta\}$ if and only if $\beta \leqslant_K \delta$, a subset $X \subseteq K$ which is both $\leqslant_K$ and D_K-closed contains $\mathscr{L}_K(X)$, hence is in $\mathscr{L}_K$. The sets in $\mathscr{L}_K$ are clearly both $\leqslant_K$ and D_K-closed.

Next we show that for every $\alpha \in K$, $\mathscr{L}_K\{\alpha\}$ is join-irreducible in $\mathscr{L}_K$: For $\hat{\alpha}$ covered in L by α we have $\mathscr{L}_K(K \cap [\hat{\alpha}]) = K \cap \mathscr{L}(K \cap [\hat{\alpha}]) \subseteq K \cap \mathscr{L}([\hat{\alpha}]) = K \cap [\hat{\alpha}]$, hence $K \cap [\hat{\alpha}] \in \mathscr{L}_K$. Clearly $K \cap [\hat{\alpha}] \neq K \cap [\alpha] = \mathscr{L}_K\{\alpha\}$ and $\mathscr{L}_K\{\beta\} \subseteq K \cap [\hat{\alpha}]$ whenever $\mathscr{L}_K\{\beta\} \subsetneqq \mathscr{L}_K\{\alpha\}$. This means that $(K, \leqslant_K)$ is order-isomorphic to the set of join-irreducible elements of $\mathscr{L}_K$ ordered by the inclusion.

It remains to show that K is standard. Let $\alpha D_K \{\alpha_1, \ldots, \alpha_n\}$ and $\mathscr{L}_K\{\alpha\} \subseteq \mathscr{L}_K([\hat{\alpha}_1] \cup \{\alpha_2, \ldots, \alpha_n\})$. Then $\alpha \in \mathscr{L}([\hat{\alpha}_1] \cup \{\alpha_2, \ldots, \alpha_n\}) \Rightarrow \mathscr{L}\{\alpha\} \subseteq \mathscr{L}([\hat{\alpha}_1] \cup \{\alpha_2, \ldots, \alpha_n\})$ a contradiction.

2.8. Theorem. *The D-admissible subsets of a standard quadricle $(J, \leqslant, D, \mathscr{L})$ form a sublattice $\mathscr{A}$ of $P(J)$, dually isomorphic to* Con (L)

in such a way that the restrictions of J *to sets in* $\mathscr{A}$ *are standard quadricles representing the corresponding quotient lattices of* L.

Proof.

(1) For $E \in \mathrm{Con}(L)$, the subset $K_E = \{\alpha \in J \mid (\alpha, \hat{\alpha}) \notin E\}$ of J is D-admissible. Assume $\alpha \leqslant \beta \vee \gamma$ and $\alpha \nleqslant \hat{\beta} \vee \gamma$ for some $\alpha \in K_E$ and $\gamma \in L$. Since $((\beta \vee \gamma) \wedge \alpha) \vee \hat{\alpha} = \alpha$, $((\hat{\beta} \vee \gamma) \wedge \alpha) \vee \hat{\alpha} = \hat{\alpha}$ and $(\alpha, \hat{\alpha}) \notin E$, it follows that $(\beta, \hat{\beta}) \notin E$, hence $\beta \in K_E$.

(2) If K is a D-admissible subset of J, then $E_K = \{(\alpha, \beta) \mid K \cap [\alpha] = K \cap [\beta]\}$ is a congruence on L (recall that $[\alpha] = \{\delta \mid \delta \leqslant \alpha$ and $\delta \in J\}$): If $K \cap [\alpha] = K \cap [\beta]$, then $K \cap [\alpha \wedge \gamma] = K \cap [\alpha] \cap [\gamma] = K \cap [\beta] \cap [\gamma] = K \cap [\beta \wedge \gamma]$. To prove $K \cap [\alpha \vee \gamma] = K \cap [\beta \vee \gamma]$, we have by 2.3 $\delta \in K \cap [\alpha \vee \gamma]$ if and only if $\delta \in K$ and $\delta\, D\, Y$ for some $Y \subseteq [\alpha] \cup [\gamma]$. Since K is D-admissible, it is $Y \subseteq K$, hence $Y \subseteq (K \cap [\alpha]) \cup (K \cap [\gamma]) \subseteq [\beta] \cup [\gamma]$ therefore $\delta \in K \cap [\alpha \vee \gamma]$ if and only if $\delta \in K \cap [\beta \vee \gamma]$.

(3) The two mappings $\varphi\colon \mathscr{A} \to \mathrm{Con}(L)\colon K \mapsto E_K$ and $\psi\colon \mathrm{Con}(L) \to \mathscr{A}\colon E \mapsto K_E$, existent by (1) and (2), are mutually inverse:

Evidently $K \subseteq K_{E_K}$. Conversely, if $\alpha \in K_{E_K}$, then $K \cap [\alpha] \neq K \cap [\hat{\alpha}]$ hence for some $\beta \in K$ it is $\beta \leqslant \alpha$ and $\beta \nleqslant \hat{\alpha}$ therefore $\alpha = \beta \in K$. Thus φ is a right inverse of ψ. But φ is injective since for $K_1, K_2 \in \mathscr{A}$, $\alpha \in K_1 \setminus K_2$ it is $(\alpha, \beta) \in E$ if and only if

$$(\alpha, \hat{\alpha}) \in E_{K_1} \setminus E_{K_2}.$$

(4) Finally, the mappings $f_K\colon L \to \mathscr{L}_K\colon \alpha \mapsto K \cap [\alpha]$, $K \in \mathscr{A}$ are epimorphisms with $\mathrm{Ker}(f_K) = E_K$, hence $L/E_K \cong \mathscr{L}_K$ and $(K, \leqslant_K, D_K, \mathscr{L}_K)$ is standard.

2.9. Corollary. *For any finite lattice* L *the ordering* $Q_D / Q_D \cap Q_D^{-1}$ *is isomorphic to the ordering of join-irreducible elements in* $\mathrm{Con}(L)$.

We see that the number of join-irreducible elements of $\mathrm{Con}(L)$ is at most equal to that of L. The equality is attained if and only if the standard quadricle J representing L is acyclic, i.e., $Q_D \cap Q_D^{-1}$ is the identity relation on J. Thus we get another characterization of $\mathfrak{F}$:

2.10. Theorem. *A lattice* L *is finitely fermentable if and only if it is finite and* L *and* $\mathrm{Con}(L)$ *have the same number of join-irreducible elements.*

2.11. Theorem. *The class* $\mathfrak{F}$ *of finitely fermentable lattices is closed under sublattices, quotient lattices and finite products of its members.*

Proof. Let $(J, \leqslant, D, \mathscr{L})$ be a standard quadricle representing a lattice $L \in \mathfrak{F}$, let L_1 be a sublattice of L and let $\mathscr{L}_1 \subseteq \mathscr{L}$ correspond to L_1 under an isomorphism of L onto $\mathscr{L}$. Form a quadricle $(J_1, \leqslant_1, D_1, \mathscr{L}'_1)$ where J_1 is the greatest element in $\mathscr{L}_1$, $\alpha \leqslant_1 \beta$ if and only if $\mathscr{L}_1\{\alpha\} \subseteq \mathscr{L}_1\{\beta\}$ and $D_1 = (D \cap J_1 \times P(J_1)) \cup \{(\alpha, \phi) \mid \alpha \in \mathscr{L}_1(\phi)\}$.

The sets from $\mathscr{L}_1$ are both $\leqslant_1$ and D_1-closed, hence $\mathscr{L}_1 \subseteq \mathscr{L}'_1$. On the other hand, any set $Y \in \mathscr{L}'_1$ is in $\mathscr{L}$, therefore for an arbitrary $X \subseteq J_1$ we have $\mathscr{L}_1(X) = \mathscr{L}\big(\bigcup_{\alpha \in X} \mathscr{L}_1\{\alpha\}\big) = \mathscr{L}\big(\bigcup_{\alpha\in X} \mathscr{L}'_1\{\alpha\}\big) \subseteq$ $\subseteq \mathscr{L}\big(\mathscr{L}'_1\big(\bigcup_{\alpha \in X} \mathscr{L}'_1\{\alpha\}\big)\big) = \mathscr{L}'_1(X)$, thus $\mathscr{L}'_1 \subseteq \mathscr{L}_1$. The quadricle $(J_1, \leqslant_1, D_1, \mathscr{L}_1)$ is clearly acyclic. We need to prove it commutable in order to conclude that L_1 is finitely fermentable:

For $X_1, \ldots, X_n \in \mathscr{L}_1$ it is $\mathscr{L}_1\big(\bigcup_{i=1}^{n} X_i\big) = \mathscr{L}\big(\bigcup_{i=1}^{n} X_i\big) =$ $= \big\{\alpha \mid \exists Y \subseteq \big(\bigcup_{i=1}^{n} X_i\big)\ (\alpha D Y)\big\} \subseteq \big\{\alpha \mid \exists Y \subseteq \big(\bigcup_{i=1}^{n} X_i\big)\ (\alpha D_1 Y)\big\}$ since from $Y \subseteq \bigcup_{i=1}^{n} X_i$ it follows $Y \subseteq J_1$, hence $\mathscr{L}_1\big(\bigcup_{i=1}^{n} X_i\big) = \big\{\alpha \mid \exists Y \subseteq$ $\subseteq \big(\bigcup_{i=1}^{n} X_i\big)\ (\alpha D_1 Y)\big\}$. Now, let $\beta \leqslant_1 \alpha$ and $\alpha D_1 X$. Then $\beta \in$ $\in \mathscr{L}_1(X) = \mathscr{L}_1\big(\bigcup_{\alpha\in X} \mathscr{L}_1\{\alpha\}\big)$, hence $\beta D_1 Y$ for some $Y \subseteq \bigcup_{\alpha \in X} \mathscr{L}_1\{\alpha\}$, i.e., $Y \prec_1 X$ for a quasiordering $\prec_1$ induced by $\leqslant_1$.

If a standard quadricle $(J, \leqslant, D, \mathscr{L})$ is acyclic then so are its restrictions to the D-admissible subsets, hence by 2.8, $\mathfrak{F}$ is closed under quotients. Finally, from the isomorphism $\mathrm{Con}(L_1 \times L_2) \cong \mathrm{Con}(L_1) \times$ $\times \mathrm{Con}(L_2)$ the closedness of $\mathfrak{F}$ under finite products immediately follows.

2.12. Remark. It is easily seen that $\mathfrak{F}$ comprises the class $\mathfrak{D}$ of all finite distributive lattices. Since $M_5 \notin \mathfrak{F}$, $\mathfrak{F}$ meets the class of modular lattices in $\mathfrak{D}$. However, $\mathfrak{F}$ is wider than $\mathfrak{D}$, e.g. $N_5 \in \mathfrak{F}$.

3. COMPLETE DISTRIBUTIVITY OF STRONGLY REPRESENTABLE LATTICES

We come back to infinite lattices to discuss the following property:

3.1. Definition. A lattice L is called *strongly representable** if for every complete lattice monomorphism $\varphi: L \to \mathrm{Eq}(A)$ it is $\mathrm{Im}(\varphi) = \mathrm{Con}(A, F)$ for a suitable system F of operations on A. Put otherwise, any complete equivalence lattice $\mathscr{L}$ isomorphic to L is a congruence lattice of some algebra.

In the finite case it is known that a lattice L is strongly representable if and only if it is distributive.

H. J. Bandelt has shown in [2] that any *completely* distributive algebraic lattice L, in the sense that for every system A of non-void subsets of L it is

$$\bigwedge\{\bigvee A \mid A \in A\} \leqslant \bigvee\{\bigwedge\{f(A)\mid A \in A\} \mid \forall A \in A \ (f(A) \in A)\},$$

is strongly representable. Our aim is to prove the converse assertion in this section. To this end, we introduce the following characterization of completely distributive algebraic lattices:

3.2. Statement. *Let L be an algebraic lattice, let K denote the set of all compact elements of L. The lattice L is completely distributive if and only if every $\alpha \in K$ can be expressed as a join $\alpha = \alpha_1 \vee \ldots \vee \alpha_n$ of compact elements $\alpha_1, \ldots, \alpha_n \in K$ so that $\alpha_i \leqslant \beta_1$ or $\alpha_i \leqslant \beta_2$ for $i = 1, \ldots, n$, whenever $\alpha \leqslant \beta_1 \vee \beta_2$.*

Proof. Let L be completely distributive. Then for $\alpha \in K$, $\alpha = \bigwedge\{\beta_1 \vee \beta_2 \mid \alpha \leqslant \beta_1 \vee \beta_2\} \leqslant \bigvee\{\bigwedge\{f(\beta_1, \beta_2) \mid \alpha \leqslant \beta_1 \vee \beta_2\} \mid f(\beta_1, \beta_2) = \beta_1$ or β_2 for $\alpha \leqslant \beta_1 \vee \beta_2\}$. By compactness of α, there exist $\alpha_1, \ldots, \alpha_n \in K$ such that $\alpha_i \leqslant \bigwedge\{f_i(\beta_1, \beta_2) \mid \alpha \leqslant \beta_1 \vee \beta_2\}$ where f_i are suitable choice

*In [5] "quasi-strongly representable" is used with the same meaning.

functions $f_i(\beta_1, \beta_2) \in \{\beta_1, \beta_2\}$ for $\alpha \leqslant \beta_1 \vee \beta_2$, providing the required decomposition $\alpha = \alpha_1 \vee \ldots \vee \alpha_n$.

Conversely, let $\alpha = \alpha_1 \vee \ldots \vee \alpha_n$ be a required decomposition of $\alpha \in K$, such that no member on the right can be cancelled. Then for any $\beta_1, \ldots, \beta_r \in L$ such that $\alpha \leqslant \beta_1 \vee \ldots \vee \beta_r$ it is $\alpha_k \leqslant \beta_1$ or $\alpha_k \leqslant \beta_2 \vee \ldots \vee \beta_r$ for $k = 1, \ldots, n$, say, $\alpha_1, \ldots, \alpha_i \leqslant \beta_1$ and $\alpha_{i+1}, \ldots, \alpha_n \leqslant \beta_2 \vee \ldots \vee \beta_r$. Thus we have $\alpha \leqslant \alpha_1 \vee \ldots \vee \alpha_i \vee \beta_2 \vee \ldots \vee \beta_r$, whence $\alpha_k \leqslant \beta_2$ or $\alpha_k \leqslant \alpha_1 \vee \ldots \vee \alpha_i \vee \beta_3 \vee \ldots \vee \beta_r$ for $k = 1, \ldots, n$. Proceeding in this way, we get, after a rearrangement of indices, $\{\alpha_1, \ldots, \alpha_j\} \prec \{\beta_1, \ldots, \beta_{r-1}\}$ and $\{\alpha_1, \ldots, \alpha_n\} \prec \{\alpha_1 \vee \ldots \vee \alpha_j, \beta_r\}$. If $\alpha_k \not\leqslant \beta_r$ for some $k > j$, then $\alpha_k \leqslant \alpha_1 \vee \ldots \vee \alpha_j$, which contradicts to the assumption that no member on the right in the equality $\alpha = \alpha_1 \vee \ldots \vee \alpha_n$ can be omitted. We have proved that $\alpha \leqslant \beta_1 \vee \ldots \vee \beta_r$ implies $\{\alpha_1, \ldots, \alpha_n\} \prec \{\beta_1, \ldots, \beta_n\}$.

Let A be a system of non-void subsets of L. If $\alpha \leqslant \bigwedge\{\bigvee A \mid A \in A\}$ for some $\alpha \in K$, i.e., $\alpha \leqslant \bigvee A$ for every $A \in A$, finite sets $\{\beta_1^A, \ldots, \beta_{r(A)}^A\} \subseteq A$, $A \in A$ can be found with $\alpha \leqslant \beta_1^A \vee \ldots \vee \beta_{r(A)}^A$. Now, α can be expressed as a join $\alpha = \alpha_1 \vee \ldots \vee \alpha_n$ of compact elements in such a way that $\{\alpha_1, \ldots, \alpha_n\} \prec \{\beta_1^A, \ldots, \beta_{r(A)}^A\}$ for every $A \in A$, which yields for every $i = 1, \ldots, n$ a choice function $f_i(A) = \beta_j^A$ with $\alpha_i \leqslant \beta_j^A$. Therefore $\alpha \leqslant \bigvee\{\bigwedge\{f(A) \mid A \in A\} \mid \forall A \in A \ (f(A) \in A)\}$, and since L is generated by K, we get the inequality of complete distributivity for L.

3.3. Theorem. *Every strongly representable algebraic lattice L is completely distributive.*

Proof. Let L be an algebraic lattice which is not completely distributive. Then by 3.2, there exists a compact element $\delta \in L$ such that for every decomposition of the form $\delta = \alpha_1 \vee \ldots \vee \alpha_n$ with $\alpha_1, \ldots, \alpha_n \in K$ some $\beta_1, \beta_2 \in L$ can be found such that $\{\alpha_1, \ldots, \alpha_n\} \not\prec \{\beta_1, \beta_2\}$. Fix some δ in L with this property and let $\gamma = \bigvee\{\alpha \in K \mid \alpha \leqslant \delta$ and $\exists \alpha_1, \ldots, \alpha_n \in K \ (\alpha = \alpha_1 \vee \ldots \vee \alpha_n)$ and $\forall \beta_1, \beta_2 \in L \ (\delta \leqslant \beta_1 \vee \beta_2 \Rightarrow \{\alpha_1, \ldots, \alpha_n\} \prec \{\beta_1, \beta_2\})\}$. Clearly, $\gamma < \delta$: If it were $\gamma = \delta$, we could express δ as $\alpha_1 \vee \ldots \vee \alpha_k$ and each α_i, $i = 1, \ldots, k$, as

$\alpha_{i1} \vee \ldots \vee \alpha_{in(i)}$ in such a way that $\bigcup_{i=1}^{k} \{\alpha_{i1}, \ldots, \alpha_{in(i)}\} \prec \{\beta_1, \beta_2\}$ for any β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$, but then $\delta = \bigvee_{i=1}^{k} (\alpha_{i1} \vee \ldots \vee \alpha_{in(i)})$ would be a decomposition of δ into a join of compact elements which would contradict to the choice of δ.

Represent L by the quadricle $(J, =, D, \mathscr{L})$, where $J = K \setminus \{0_L\}$ and $D = \{(\alpha, \{\alpha_1, \alpha_2\}) \mid \alpha \leqslant \alpha_1 \vee \alpha_2\}$. By 1.15, we can form two yeast J-graphs (A, r, v) and (B, s, w) in the systems of cells $\mathfrak{C}_J(2)$ and $\mathfrak{C}_J(5)$ creating two distinct representations $\psi\colon \mathscr{L} \to \mathrm{Con}(A)$ and $\chi\colon \mathscr{L} \to \mathrm{Con}(B)$, by congruences of the associated algebras, respectively. The two yeast graphs A, B can so be chosen that $|A \cap B| = 1$ and an equivalence representation $\varphi\colon \mathscr{L} \to \mathrm{Eq}(A \cup B)$ can be defined by $\varphi(X) = \mathscr{E}_{A \cup B}(\psi(X) \cup \chi(X)) = \psi(X) \cup \chi(X) \cup (\psi(X) \circ \chi(X)) \cup (\chi(X) \circ \psi(X))$ for $X \in \mathscr{L}$.

Our aim is to show that $\mathrm{Im}(\varphi)$ is not a congruence lattice. To this end, define a relation $R_\delta = \bigcap_{\delta \leqslant \beta_1 \vee \beta_2} R_{\beta_1 \beta_2}$, where $R_{\beta_1 \beta_2} = \varphi(\mathscr{L}\{\beta_1\}) \circ \varphi(\mathscr{L}\{\beta_2\}) \circ \varphi(\mathscr{L}\{\beta_1\}) \circ \varphi(\mathscr{L}\{\beta_2\})$ and denote $E = \mathscr{E}_{A \cup B}(R_\delta)$. Clearly, E will be in any congruence lattice on $A \cup B$ containing $\mathrm{Im}(\varphi)$. The remaining part of the proof shows that $E \notin \mathrm{Im}(\varphi)$ by proving that there is no $X \in \mathscr{L}$ with $\varphi(X) = E$:

It is readily seen, from the definition of φ, that $\varphi(X) \cap A^2 = \psi(X)$, $\varphi(X) \cap B^2 = \chi(X)$, for $X \in \mathscr{L}$.

Let c denote the unique common point of A and B. All equivalencies in $\mathrm{Im}(\varphi)$ are stable under the transformation $f, g\colon A \cup B \to A \cup B$ defined by

$$f(a) = \begin{cases} a & \text{for } a \in A \\ c & \text{for } a \in B, \end{cases} \qquad g(b) = \begin{cases} b & \text{for } b \in B \\ c & \text{for } b \in A, \end{cases}$$

hence $f(R_\delta) = R_\delta \cap A^2$ and $g(R_\delta) = R_\delta \cap B^2$ therefore $E \cap A^2 = \mathscr{E}_A(R_\delta \cap A^2)$ and $E \cap B^2 = \mathscr{E}_B(R_\delta \cap B^2)$.

If an edge $x \in r$ in the yeast graph (A, r, v) is valuated $v(x) \leqslant \delta$, then for any β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$ there is a path x_1, x_2, x_3, x_4 in r valuated $(\nu(x_1), \nu(x_2), \nu(x_3), \nu(x_4)) = (\beta_1, \beta_2, \beta_3, \beta_4)$, which means that $x \in R_\delta$. Now, for $(a, b) \in \psi(\mathscr{L}\{\delta\}) = \mathscr{E}_A \nu^{-1}(\mathscr{L}\{\delta\})$, there is a path $y_1, \dots, y_m$ in r connecting a to b such that $\nu(y_i) \leqslant \delta$ for $i = 1, \dots, m$ therefore $(a, b) \in \mathscr{E}_A(R_\delta \cap A^2)$, which proves the inclusion $\psi(\mathscr{L}\{\delta\}) \subseteq E \cap A^2$.

Let $(a, b) \in R_{\beta_1\beta_2} \cap B^2$ for some β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$, i.e., there exists a sequence $a = c_0, c_1, \dots, c_n = b$ such that $x_i = (c_{i-1}, c_i) \in \in s$ for $i = 1, \dots, n$ and, for some $1 \leqslant p \leqslant q \leqslant r \leqslant n$, $x_1, \dots, x_p \in \in \varphi(\mathscr{L}\{\beta_1\})$, $x_{p+1}, \dots, x_q \in \varphi(\mathscr{L}\{\beta_2\})$, $x_{q+1}, \dots, x_r \in \varphi(\mathscr{L}\{\beta_1\})$, $x_{r+1}, \dots, x_n \in \varphi(\mathscr{L}\{\beta_2\})$. Let us call every sequence of this kind a $\beta_1\beta_2$-sequence for (a, b) in B. If a path $x_1, \dots, x_n$ determined by some $\beta_1\beta_2$-sequence $c_0, c_1, \dots, c_n$ contains a segment $x_{i+1}, \dots, x_{i+10}$ connecting the endpoints of an edge $x \in s$ in the cell with the base x, then $x \in \varphi(\mathscr{L}\{\beta_1\}) \cup \varphi(\mathscr{L}\{\beta_2\})$, since assuredly there are two adjacent edges in $x_{i+1}, \dots, x_{i+10}$ both contained either in $\varphi(\mathscr{L}\{\beta_1\})$ or in $\varphi(\mathscr{L}\{\beta_2\})$, hence $w(x) \leqslant \beta_1$ or $w(x) \leqslant \beta_2$. Consequently, we can replace $x_{i+1}, \dots, x_{i+10}$ by a single edge x, getting thus a new path $x_1, \dots, x_i, x, x_{i+11}, \dots, x_n$ corresponding to the $\beta_1\beta_2$-sequence $c_0, c_1, \dots, c_i, c_{i+10}, \dots, c_n$. If $a, b \in B_n$, where $B_n = (B_n, s_n, w_n)$ denotes the n-th growth of the yeast graph B, the above procedure of "strightening" $\beta_1\beta_2$-sequencies enables us to find a $\beta_1\beta_2$-sequence for (a, b) contained entirely in B_n.

Next we shall show that for every $(a, b) \in R_\delta \cap B^2$ there exists a sequence $a = c_0, c_1, \dots, c_m = b$ such that $\{w(c_0, c_1), \dots, w(c_{m-1}, c_m)\} \prec \prec \{\beta_1, \beta_2\}$ for every β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$, by an inductive argument on the minimal n such that $a, b \in B_n$: For $n = 0$ it is true, since (B_0, s_0) is a tree, hence for any two points $a, b \in B_0$ there is a unique path from a to b in B_0. Assume further the assertion is true for some $n \geqslant 0$, let $a, b \in B_{n+1}$, and consider three cases:

(1) Let $a \in B_{n+1} \setminus B_n$ and $b \in B_n$, let $\{c, d\} \subseteq B_n$ be the base of the cell of $(n+1)$-th generation containing a and let there be at most five edges between a and c. For any β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$, we

have $(a, b) \in R_{\beta_1\beta_2}$ hence there is a $\beta_1\beta_2$-sequence for (a, b) contained entirely in B_{n+1}. If this sequence contains c, then all the edges between a and c are valued $\leqslant \beta_1$ or β_2, if it contains d, then since there are at least five edges between a and d, there must be two adjacent edges valued both $\leqslant \beta_1$ or β_2, hence again also the edges between a and c are valued $\leqslant \beta_1$ or β_2, which proves the stated property for (a, c). Applying induction hypothesis to (c, b), we obtain a path from a to b with its edges valued $\leqslant \beta_1$ or β_2 for any β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$.

(2) Let $a, b \in B_{n+1} \setminus B_n$ and not both in the same cell of $(n+1)$-th generation, let c and d be points in B_n such that there are paths from a to c and from b to d of length $\leqslant 5$. By a similar argument, these shortest paths from a, b to B_n are valued or $\leqslant \beta_1$ or β_2 for any β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$ and to (c, d) the induction hypothesis applies.

(3) Let $a, b \in B_{n+1} \setminus B_n$ and let a and b be both contained in the same path connecting the end-points of some edge $\{c, d\} \in s_n$. Again, the shortest path between a and b has the desired property.

Now we are able to prove the inclusion $R_\delta \cap B^2 \subseteq \chi(\mathscr{L}\{\gamma\})$: If $(a, b) \in R_\delta \cap B^2$, then there exists a path $x_1, \ldots, x_n$ in s from a to b with $\{w(x_1), \ldots, w(x_n)\} \prec \{\beta_1, \beta_2\}$ for any β_1, β_2 with $\delta \leqslant \beta_1 \vee \beta_2$, hence

$$(a, b) \in \chi(\mathscr{L}\{w(x_1) \vee \ldots \vee w(x_n)\}) \subseteq \chi(\mathscr{L}\{\gamma\}).$$

Consequently, $E \cap B^2 \subseteq \chi(\mathscr{L}\{\gamma\})$, which together with $E \cap A^2 \supseteq \psi(\mathscr{L}\{\delta\})$ and $\gamma < \delta$ shows that $E \notin \mathrm{Im}(\varphi)$ and we are done.

4. CHARACTERIZATION OF EQUIVALENCE LATTICES AND THEIR PRODUCTS

In this section a characterization of equivalence lattices and their products will be given.

To every equivalence lattice $\mathrm{Eq}(U)$ a quadricle $(J, =, R, \mathrm{Eq}(U))$ can be assigned, where J is a set of all subsets of U with exactly two points and

$$R = \{(\{a, b\}, \{\{b, c\}, \{a, c\}\}) \mid a, b, c \text{ are distinct elements of } U\} \cup$$
$$\cup \{(\{a, b\}, \{\{a, b\}\}) \mid a, b \text{ are distinct elements of } U\}.$$

The relation R has an exchange property in this sense: if $x\,R\,\{y, z\}$ then also $y\,R\,\{x, z\}$ and $z\,R\,\{x, y\}$. It is easy to see, that the relation R (and so the whole quadricle) can be fully defined by a system T of three-point subsets of J. This is a motivation for the following definition.

4.1. Definition. We shall call a triple $(J, S, \mathscr{L})$ a *dependence system* if J is a set, $S \subseteq P(J)$ and $(J, =, R, \mathscr{L})$ is a quadricle with R defined by

$$R = \{(x, \{y_1, \ldots, y_n\}) \mid \{x, y_1, \ldots, y_n\} \in S\} \cup \{(x, \{x\}) \mid x \in J\}\}.$$

Our main goal in this section is to prove necessary and sufficient conditions for S consisting from three-point subsets of some set J to give rise to a dependence system $(J, S, \mathscr{L})$ with $\mathscr{L}$ isomorphic to some equivalence lattice or to a product of equivalence lattices. As a corollary of these, we shall obtain necessary and sufficient conditions (local ones) for a lattice L to be isomorphic to some product of equivalence lattices.

For a given set J and a natural number n, let $P_n(J)$ denote the set of all subsets of J of cardinality n.

4.2. Note. If $(J_i, S_i, \mathscr{L}_i)$ are dependence systems for $i \in I$, and J_i are mutually disjoint, then $\left(\bigcup_{i \in I} J_i, \bigcup_{i \in I} S_i, \prod_{i \in I} \mathscr{L}_i\right)$ is also a dependence system.

4.3. Theorem. *Let* $(J, S, \mathscr{L})$ *be a dependence system* $S \subseteq P_3(J)$. *Then* $\mathscr{L}$ *is isomorphic to some equivalence lattice* Eq (U) *if and only if the following five conditions are satisfied:*

EC 1: *If* $A, B \in S$, $A \neq B$, *then* $\operatorname{card}(A \cap B) \leqslant 1$;

EC 2: *if* $\{a, b, c\}$, $\{c, d, e\}$ *are distinct elements of* S, *then there exists* $A \in S$ *such that* $\{a, d\} \subseteq A$ *iff* $\{a, e\} \nsubseteq B$ *for any* $B \in S$;

EC 3: *if* $\{a, b, c\}$, $\{c, d, e\}$, $\{e, f, a\}$ *are distinct elements of* S, *then* $\{b, d, f\} \in S$;

EC 4: *if* a, b, c, d, e, f, g, h *are distinct points of* J *and* $\{a, b, c\}, \{c, d, e\}, \{e, f, g\}, \{g, h, a\} \in S$ *then there exists* $i \in I$ *such that* $\{c, i, g\} \in S$;

EC 5: *if* $a, b \in J$, $a \neq b$, *then there exist* $A, B \in S$ *such that* $a \in A$, $b \in B$, $A \cap B \neq \phi$.

$\mathscr{L}$ *is isomorphic to some product* $\prod_{i \in I} \mathrm{Eq}(U_i)$ *of equivalence lattices if and only if* S *satisfies conditions* EC 1, EC 2, EC 3, EC 4.

Proof.

(a) Let φ be an isomorphism of $\mathscr{L}$ onto $\mathrm{Eq}(U)$. Since $S \subseteq P_3(J)$, we have $\phi \in \mathscr{L}$ and $\{a\} \in \mathscr{L}$ for all $a \in J$. Hence all $\{a\}$ are atoms in $\mathscr{L}$ and $\varphi(\{a\})$ are atoms in $\mathrm{Eq}(U)$.

Let $(P_2(U), T, \mathrm{Eq}(U))$ be the above described dependence system. We can define a map $\omega: J \to P_2(U)$ by $\omega(a) = \{u; v\}$ if and only if $\varphi(\{a\})$ identifies u, v. Then it is easy to see that ω is one-to-one.

Since φ is an isomorphism, $\{a, b, c\} \in S$ if and only if $\{\omega(a), \omega(b), \omega(c)\} \in T$. So ω is an isomorphism of the set system (J, S) onto $(P_2(U), T)$. But T satisfies conditions EC 1-EC 5, hence so does S.

If φ is an isomorphism of $\mathscr{L}$ onto $\prod_{i \in I} \mathrm{Eq}(U_i)$ we can assume that U_i are mutually disjoint. Then ω is an isomorphism of the set systems (J, S) and $\left(\bigcup_{i \in I} P_2(U_i), \bigcup_{i \in I} T_i\right)$. But $\bigcup_{i \in I} T_i$ satisfies conditions EC 1-EC 4 and so does S.

(b) Let $(J, S, \mathscr{L})$ be a dependence system satisfying the conditions EC 1-EC 5. Let $\mathfrak{N}$ be a system of subsets $N \subseteq J$ enjoying the following property:

(*) if $a, b \in N$, $a \neq b$ then there exists $c \in J \setminus N$ such that $\{a, b, c\} \in S$

$\mathfrak{N}$ is non-void since $\phi \in \mathfrak{N}$.

If $\mathfrak{M} \subseteq \mathfrak{N}$ is a chain (ordered linearly by inclusion), then also $\bigcup \mathfrak{M} \in \mathfrak{N}$.

By the Zorn's lemma, there exists a maximal set $M \in \mathfrak{N}$. We shall prove that $\mathscr{L}(M) = J$.

Let $x \in J \setminus \mathscr{L}(M)$. If $M = \phi$, then $\{x\} \in \mathfrak{N}$ and M is not maximal in $\mathfrak{N}$. Let $a \in M$. Then by EC 5 we have $A, B \in S$, $a \in A$, $x \in B$ and $A \cap B \neq \phi$. If $A \cap (J \setminus \mathscr{L}(M)) \neq \phi$ we have the existence of $a \in M$ $x \in J \setminus \mathscr{L}(M)$ and $A \in S$ such that $\{a, x\} \subseteq A$.

If $A \subseteq \mathscr{L}(M)$, then $B \cap \mathscr{L}(M) \neq \phi \neq B \cap (J \setminus \mathscr{L}(M))$. If $B = \{b, c, x\}$ and $b \in B \cap \mathscr{L}(M)$ then $c \notin \mathscr{L}(M)$ since $\mathscr{L}(M)$ is closed in $\mathscr{L}$. But by condition EC 2 there exists $C \in S$ such that $\{a, c\} \subseteq C$ or $\{a, x\} \subseteq C$.

We can conclude: if $J \setminus \mathscr{L}(M) \neq \phi$ then there exist $a_1 \in M$, $x \in J \setminus \mathscr{L}(M)$ and $A \in S$ such that $\{a_1, x\} \subseteq A$.

Let $a_2 \neq a_1, a_2 \in M$. Then by $(*)$ there exists $c_1 \in \mathscr{L}(M) \setminus M$ such that $\{a_1, a_2, c_1\} \in S$. If $A = \{a_1, x, y\}$ $(y \in J \setminus \mathscr{L}(M))$, then by EC 2 there exists $z \in J - \mathscr{L}(M)$ such that $\{a_2, x, z\} \in S$ or $\{a_2, y, z\} \in S$. Let for example $\{a_2, x, z\} \in S$. If $a_3 \in M$, $a_3 \neq a_1, a_2$, then there exist $c_2, c_3 \in \mathscr{L}(M) - M$ such that (by $(*)$) $\{a_1, a_3, c_2\} \in S$ and $\{a_2, a_3, c_3\} \in S$. All points $a_1, a_2, a_3, c_2, c_3, x, y, z$ are distinct and $\{a_1, c_2, a_3\}$, $\{a_3, c_3, a_2\}$, $\{a_2, z, x\}$, $\{x, y, a_1\} \in S$. Then by EC 4 there exists $i \in J$ such that $\{a_3, i, x\} \in S$. Since $x \notin \mathscr{L}(M)$, it is $i \notin M$. The set $M \cup \{x\}$ then satisfies condition $(*)$, which contradicts to the maximality of M.

Let $M = \{a_i\}_{i \in I}$, $\sigma \notin I$ be an arbitrary point and $(P_2(I \cup \{\sigma\}), T, \mathrm{Eq}\,(I \cup \{\sigma\}))$ be the above described dependence system. We can define a map $\varphi: P_2(I \cup \{\sigma\}) \to J$ by

$$\varphi(\{\sigma, i\}) = a_i \quad \text{for} \quad i \in I$$

$$\varphi(\{i, j\}) = a_{ij} \quad \text{for} \quad i \neq j, \ i, j \in I,$$

where $a_{ij} = a_{ji}$ is a uniquely defined (by EC 1) point from J such that $\{a_i, a_j, a_{ij}\} \in S$.

Since $a_{ij} \neq a_{kl}$ if $\{i, j\} \neq \{k, l\}$, φ is one-to-one map to some subset J' of J.

From the definition of φ, we have that $\{\varphi(\{\sigma, i\}), \varphi(\{\sigma, j\}), \varphi(\{i, j\})\} \in$ $\in S$ for $i, j \in I$, $i \neq j$.

Since $\{a_i, a_{ij}, a_j\}$, $\{a_j, a_{jk}, a_k\}$, $\{a_k, a_{ik}, a_i\}$ belong to S and are distinct if i, j, k are distinct, we have by EC 3 that also $\{a_{ij}, a_{jk}, a_{ik}\} \in S$. Hence $\{\varphi(a), \varphi(b), \varphi(c)\} \in S$ if $\{a, b, c\} \in T$.

It is $\{a_i, a_j, a_{ij}\} \in S$, $\{a_j, a_k, a_{jk}\} \in S$, $\{a_i, a_k, a_{ik}\} \in S$ and, if i, j, k are distinct, it cannot be (by EC 2) $\{a_i, a_{jk}\} \subseteq C$ for any $C \in S$.

If $\{a_{ij}, a_{kl}, b\} \in S$ for some distinct $i, j, k, l \in I$ then $\{a_i, a_j, a_{ij}\}$, $\{a_{ij}, a_{kl}, b\}$ are distinct elements of S. By EC 2, there exists $c \in J$ such that $\{a_i, c, a_{kl}\} \in S$ or $\{a_i, c, b\} \in S$. But we have just proved that the first case is impossible and from the second one, using condition EC 3, it follows $\{a_j, a_{kl}, c\} \in S$ which is, by the same reason, also impossible.

So we have proved that if $\{a, b, c\} \in S$ and card $(\{a, b, c\} \cap \varphi(P_2(I \cup \{\sigma\}))) \geqslant 2$, then $\{a, b, c\} = \{a_i, a_j, a_{ij}\}$ or $\{a, b, c\} = \{a_{ij}, a_{jk}, a_{ik}\}$ for some $i, j, k \in I$.

From this it follows that $\varphi(P_2(I \cup \{\sigma\})) \in \mathscr{L}$ and if $\{\varphi(a), \varphi(b), \varphi(c)\} \in S$ for some $a, b, c \in P_2(I \cup \{\sigma\})$ then $\{a, b, c\} \in T$.

Since $M \subseteq \varphi(P_2(I \cup \{\sigma\}))$, then $J = \mathscr{L}(M) = \varphi(P_2(I \cup \{\sigma\}))$ and then φ is onto. So we have proved that φ is an isomorphism of the set systems $(P_2(I \cup \{\sigma\}), T)$ and (J, S). Thus $\mathscr{L}$ is isomorphic to Eq $(I \cup \{\sigma\})$.

(c) Let us assume, that S satisfies conditions Ec 1-EC 4. Let $\sim$ be a relation on J defined by: $a \sim b$ iff $a = b$ or a, b satisfy condition EC 5. To prove that $\sim$ is an equivalence it is sufficient to show that $\sim$ is transitive.

If $a \sim b$, $b \sim c$ and $a = b$ or $b = c$ then also $a \sim c$.

Let a, b, c be distinct points of J. We shall show that if $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ are points of J and $\{a_0, a_1, a_2\} \in S$, $\{a_2, a_3, a_4\} \in S$, $\{a_4, a_5, a_6\} \in S$, then there exist points $b_0 = a_0$, $b_1, b_2, b_3, b_4 = a_6$ from J such that $\{b_0, b_1, b_2\} \in S$ and $\{b_2, b_3, b_4\} \in S$. We can assume, that points $a_0, a_1, \ldots, a_6$ are distinct. Then, according to EC 2, there exists $c_1 \in J$ such that $\{a_0, c_1, a_3\} \in S$ or $\{a_0, c_1, a_4\} \in S$. If $\{a_0, c_1, a_4\} \in S$, we can take $b_0 = a_0$, $b_1 = c_1$, $b_2 = a_4$, $b_3 = a_5$, $b_4 = a_6$. If $\{a_0, c_1, a_3\} \in S$, then by EC 3 it is also $\{a_1, c_1, a_4\} \in S$. But according to EC 2, there exists $c_2 \in J$ such that $\{c_1, c_2, a_5\} \in S$ or $\{c_1, c_2, a_6\} \in S$. If $\{c_1, c_2, a_6\} \in S$, we can take $a_0 = b_0$, $b_1 = a_3$, $b_2 = c_1$, $b_3 = c_2$ and $b_4 = a_6$. If $\{c_1, c_2, a_5\} \in S$, by EC 3 it is $\{a_1, c_2, a_6\} \in S$ and we can take $b_0 = a_0$, $b_1 = a_2$, $b_2 = a_1$, $b_3 = c_2$, $b_4 = a_6$.

Now if $a \sim b$, $b \sim c$, $a \neq b \neq c$ then there exist points $a = a_0$, $a_1, a_2, a_3, a_4 = b$, $a_5, a_6, a_7, a_8 = c$ such that $\{a_0, a_1, a_2\} \in S$, $\{a_2, a_3, a_4\} \in S$, $\{a_4, a_5, a_6\} \in S$ and $\{a_6, a_7, a_8\} \in S$.

Using the property of S just proved we can conclude that $a \sim c$.

Now, if $\{J_i\}_{i \in I}$ are all classes of equivalence $\sim$, the dependence systems $\{J_i, S \cap P_3(J_i), \mathscr{L} \cap P(J_i)\}$ satisfy conditions EC 1-EC 5, hence by the part (b) of the proof, $\mathscr{L} \cap P(J_i)$ are isomorphic to some Eq (U_i). Since J_i are mutually disjoint, it is $(J, S, \mathscr{L}) = \left(\bigcup_{i \in I} J_i, \bigcup_{i \in I} S \cap P_3(J_i), \mathscr{L} \right)$, and, by 4.2, $\mathscr{L}$ is isomorphic to $\prod_{i \in I} \mathrm{Eq}\,(U_i)$.

4.4. Lemma. *Let* $[E, F]$ *be an interval in* Eq (A). *Then* $[E, F]$ *is isomorphic to some product of equivalence lattices.*

Proof. Let Φ be a partition of A into the classes of equivalence F. For every $B \in \Phi$, let us denote $\bar{B} = B / E \cap B^2$, i.e., $\bar{B}$ is the set of the classes of the equivalence E, which are contained in B. Then $[E, F]$ is isomorphic to the product $\prod_{B \in \Phi} \mathrm{Eq}\,(\bar{B})$. The isomorphism is given by $\mathscr{C} \in [E, F]$, $\mathscr{C} \rightarrow \{\mathscr{C}_B\}_{B \in \Phi} \in \prod_{B \in \Phi} \mathrm{Eq}\,(\bar{B})$, where for $x, y \in \bar{B}$ it is $x\ {}_B\, y$ iff for every $a, b \in A$, $a \in x$ and $b \in y$ implies $a\, \mathscr{C}\, b$.

Now, let us recall some theorems of the theory of semimodular and geometric lattices.

4.5. Theorem. *In an atomic semimodular lattice an element* a *covers* b *iff there exists an atom* c, $c \nleq b$ *such that* $a = b \vee c$.

4.6. Theorem. *If* L *is a finite atomic semimodular lattice and* $\mathscr{L}$ *is the corresponding closure operator on the set* J *of all atoms of* L, *then* $\mathscr{L}$ *has the following exchange property: if* $a, b \in J$, $A \subseteq J$, $a \notin \mathscr{L}(A)$ *and* $a \in \mathscr{L}(A \cup \{b\})$, *then also* $b \in \mathscr{L}(A \cup \{a\})$.

The proofs of Theorems 4.5 and 4.6 are easy and can be found in [3] or [4].

4.7. Theorem. *A finite lattice* L *is isomorphic to some product of equivalence lattices iff the following conditions hold:*

DC 1: L *is atomic and semimodular;*

DC 2: *the diagram of* L *does not contain subdiagram given by Fig.* 1;

DC 3: *any subdiagram of the diagram of* L *of the form given by Fig.* 2 *can be completed to one and only one of the diagrams given by Fig.* 3a *and* 3b;

DC 4: *any subdiagram of diagram of* L *of the form given by Fig.* 4 *can be completed to the diagram of Fig.* 5;

DC 5: *any subdiagram of the diagram of* L *of the form given by Fig.* 6 *can be completed to the diagram of Fig.* 7;

DC 6: *any subdiagram of the diagram of* L *of the form given by Fig.* 8 *can be completed to one and only one of the diagrams of Fig.* 9a *and* 9b.

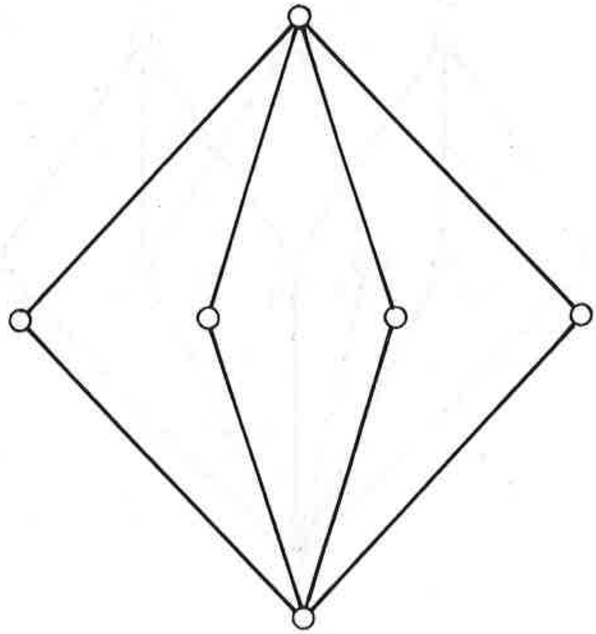

Fig. 1

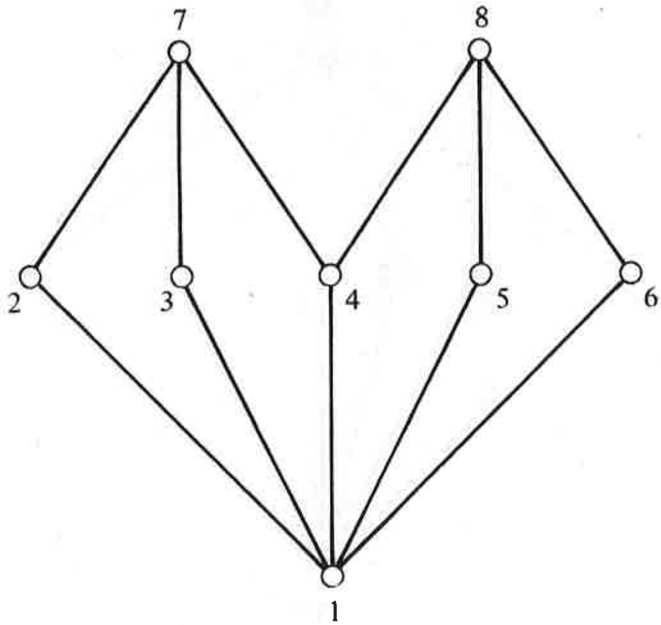

Fig. 2

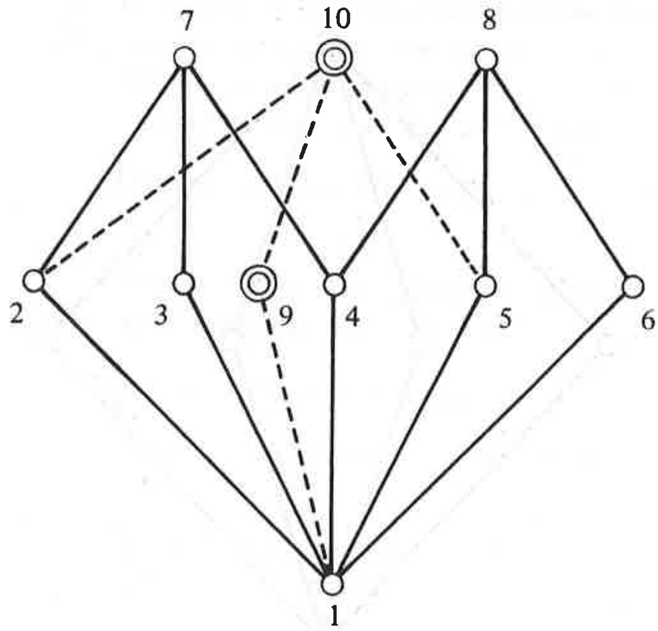

Fig. 3a

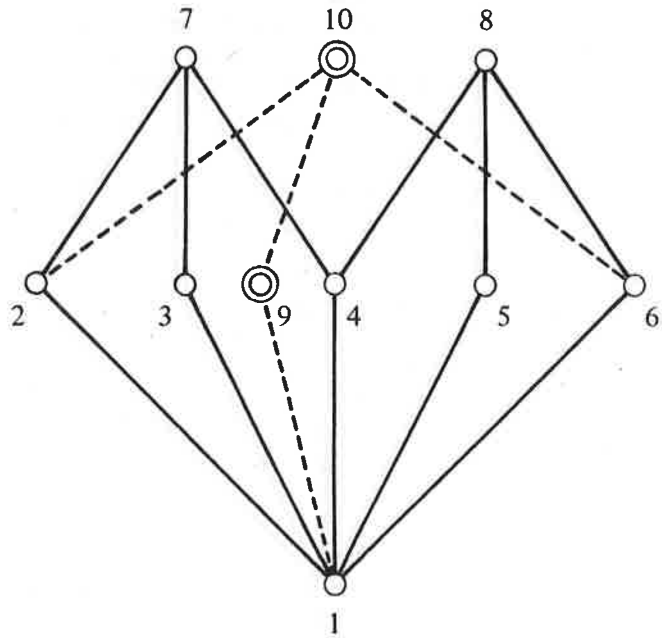

Fig. 3b

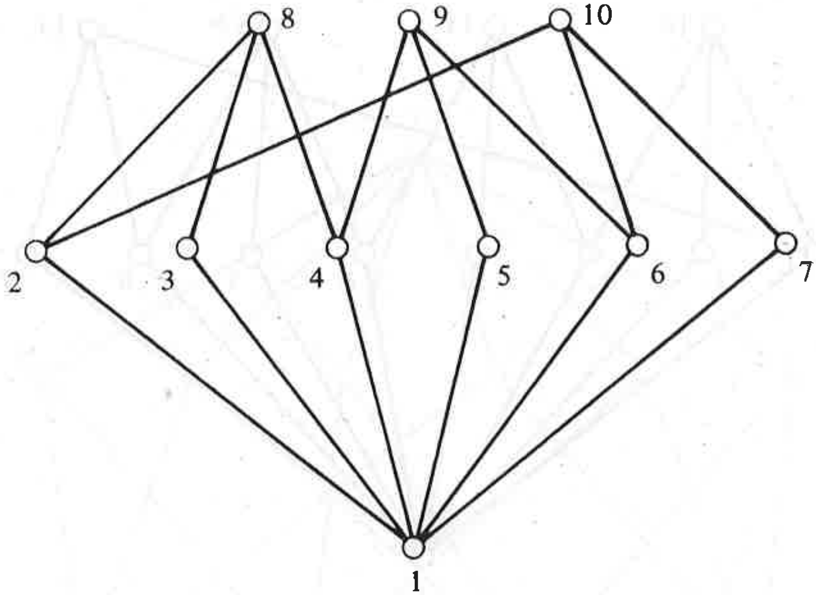

Fig. 4

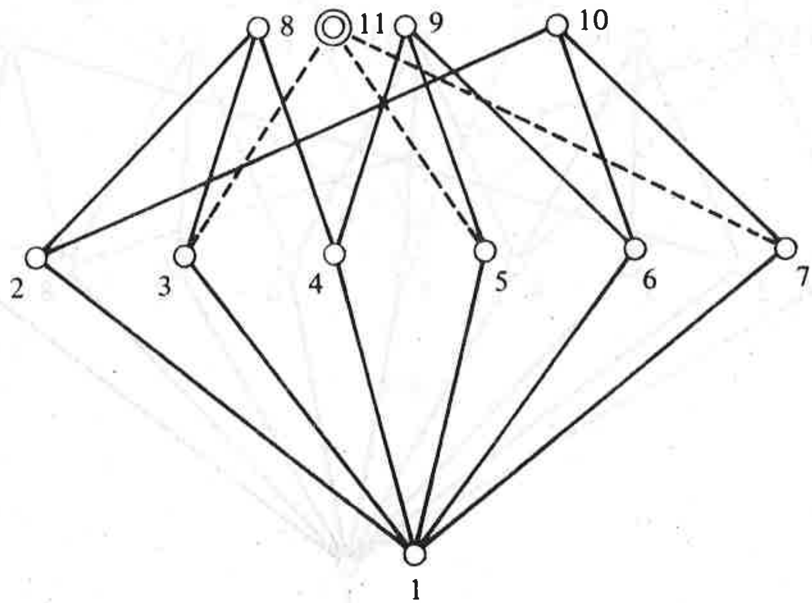

Fig. 5

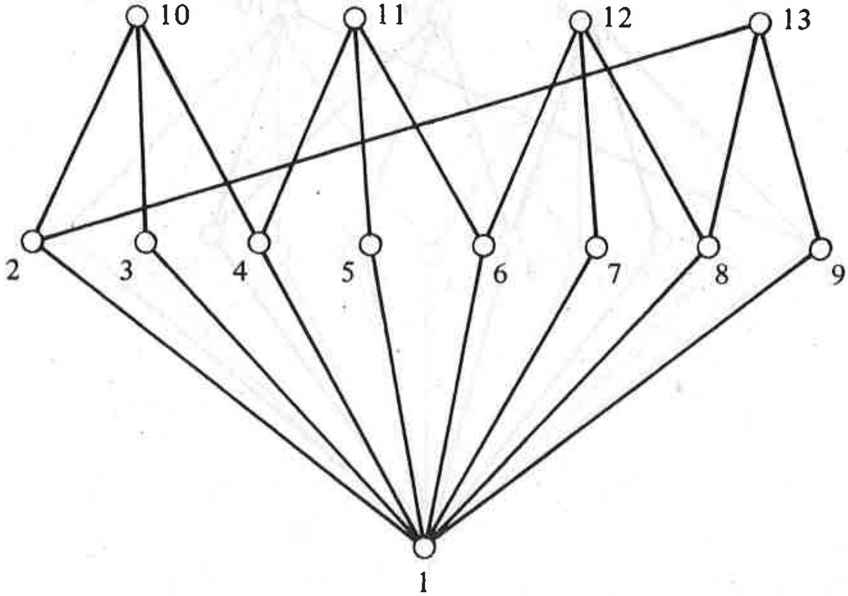

Fig. 6

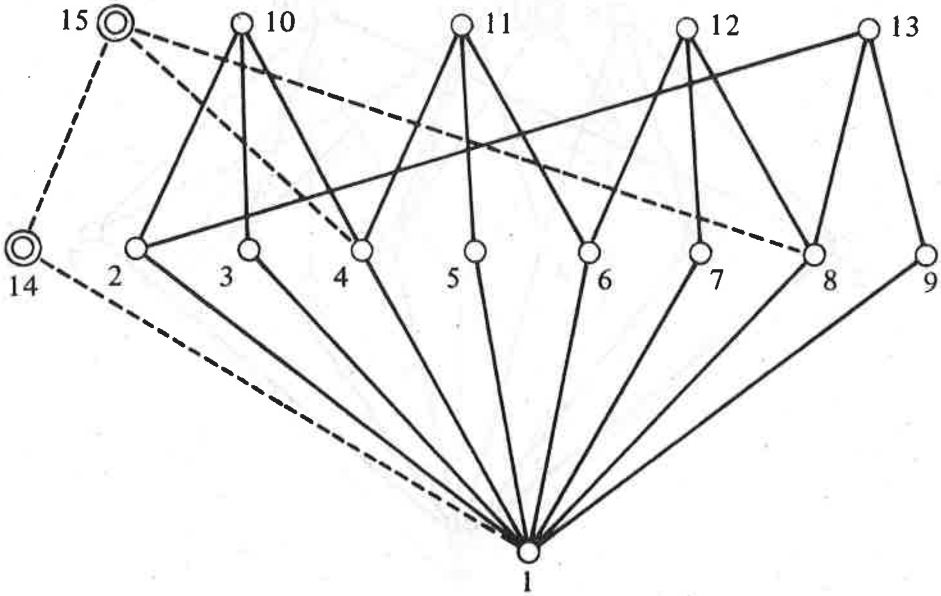

Fig. 7

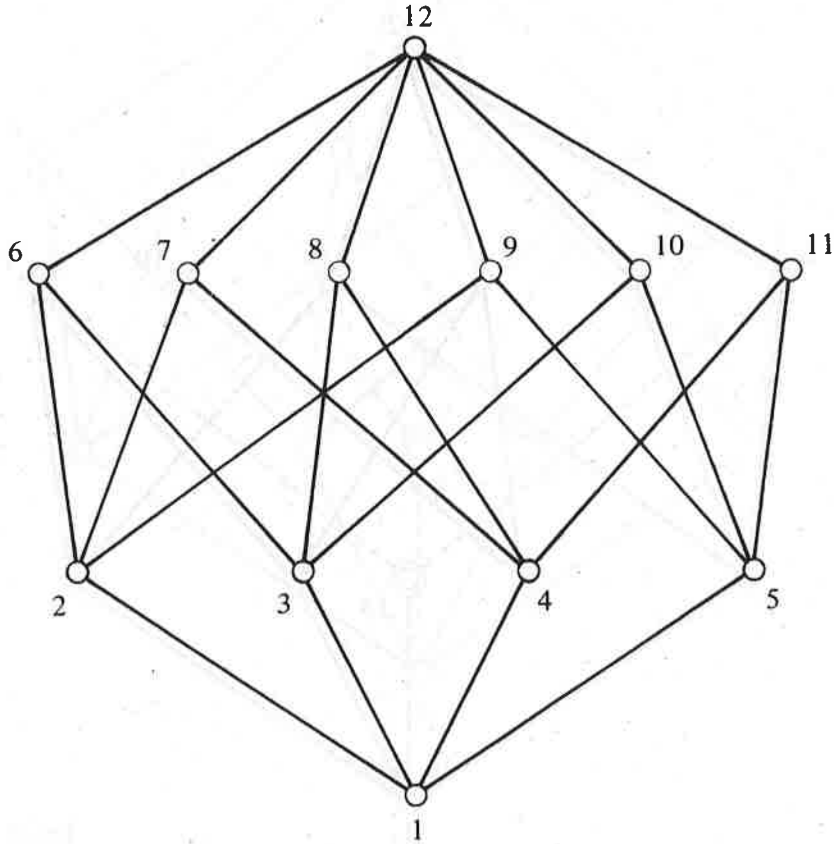

Fig. 8

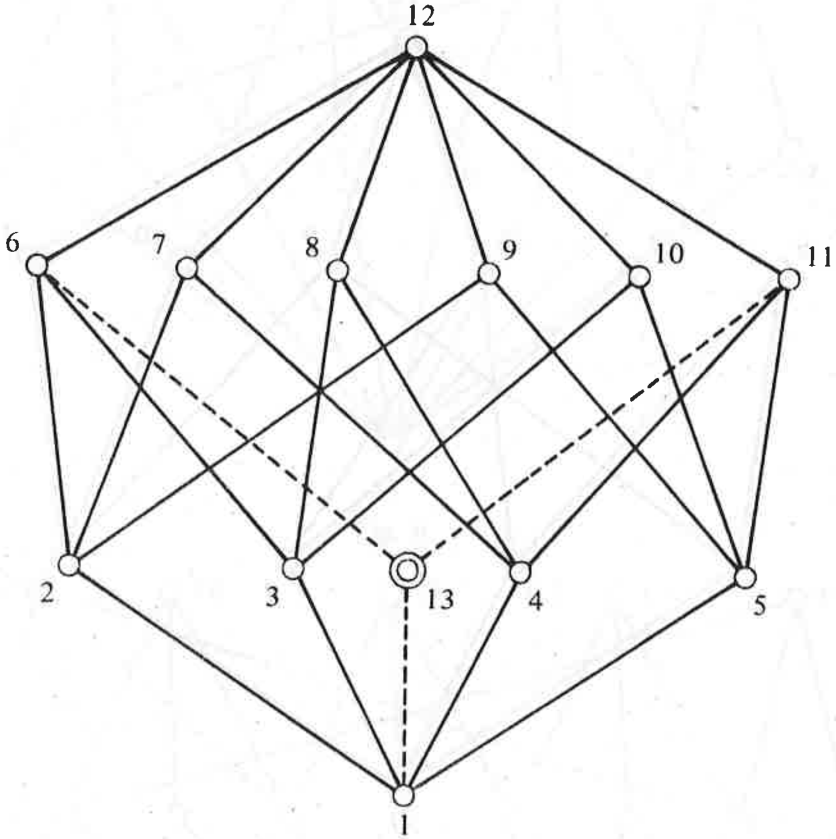

Fig. 9a

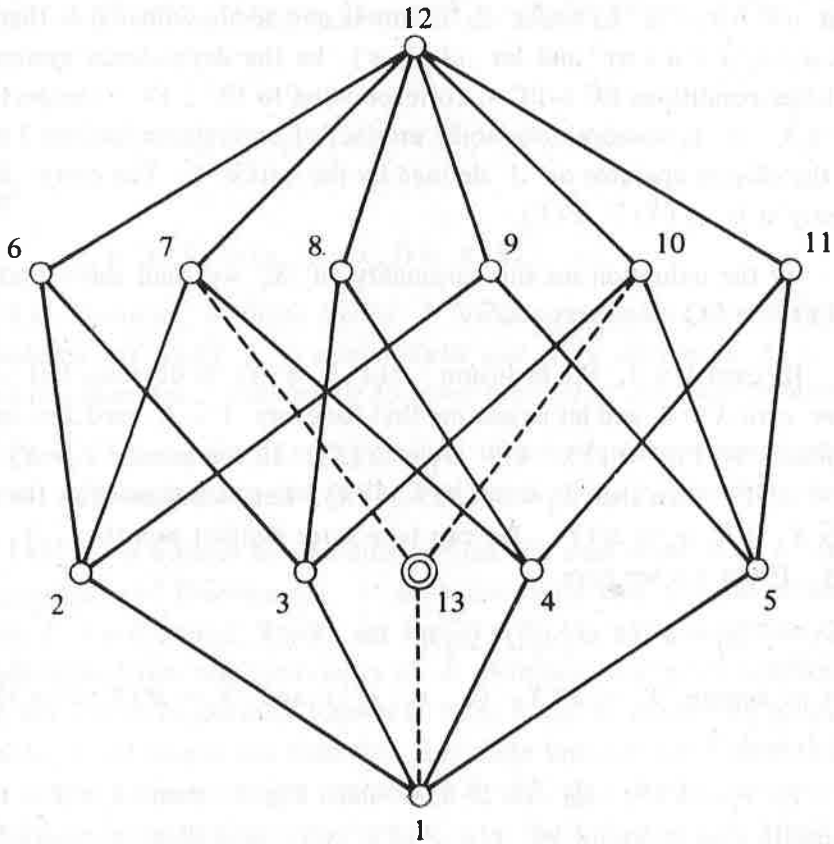

Fig. 9b

Proof.

(a) We shall prove that conditions DC 1-DC 6 are necessary. Condition DC 1 is a well known theorem from the theory of equivalence lattices (see [4]). We can assume that 1 in the diagrams of Figures 2, 4, 6, 8 corresponds to the identity on some set and the points which cover 1 correspond to the equivalences which have one two-point class and remaining classes contain only one point. The verification of conditions DC 2-DC 6 is now easy.

(b) Let conditions DC 1-DC 6 be satisfied by L. Let J denote the set of all atoms of L, S be the set of all sets $\{a, b, c\} \in P_3(J)$ such that $a < b \vee c$ in L (since L is atomic and semimodular it is then also $b < a \vee c$, $c < a \vee b$) and let $(J, S, \mathscr{K})$ be the dependence system. S satisfies conditions EC 1-EC 4, corresponding to DC 2-DC 5, respectively. By 4.3, $\mathscr{K}$ is isomorphic to some product of equivalence lattices. Let $\mathscr{L}$ be the closure operator on J defined by the lattice L. For every $X \subseteq J$, clearly it is $\mathscr{K}(X) \subseteq \mathscr{L}(X)$.

By the induction on the cardinality of X, we shall show that also $\mathscr{L}(X) \subseteq \mathscr{K}(X)$ for every $X \subseteq J$.

If $\operatorname{card} X \leqslant 2$, the inclusion $\mathscr{L}(X) \subseteq \mathscr{K}(X)$ is obvious. Let $X \subseteq J$ have $\operatorname{card} X > 2$ and let us assume that for every $Y \subseteq J$, $\operatorname{card} Y < \operatorname{card} X$ implies $\mathscr{L}(Y) = \mathscr{K}(Y)$. Let $x_1 \in \mathscr{L}(X)$. If for some $Y \subsetneqq X$ it is $x_1 \in \mathscr{L}(Y)$ then also $x_1 \in \mathscr{K}(Y) \subseteq \mathscr{K}(X)$. Let us assume that for every $Y \subsetneqq X$ it is $x_1 \notin \mathscr{L}(Y)$. We can take three distinct points $x_2, x_3, x_4 \in$ $\in X$. From 4.6 we have

$$(**) \qquad x_i \in \mathscr{L}((X \cup \{x_1\}) \setminus \{x_i\}) \quad \text{for} \quad i = 1, 2, 3, 4.$$

Let us denote $Y_0 = \mathscr{L}(X - \{x_2, x_3, x_4\})$ and $Y_i = \mathscr{L}(Y_0 \cup \{x_i\})$ for $i = 1, 2, 3, 4$.

If $x_i \in \mathscr{L}(Y)$ for $i = 2, 3, 4$ and $Y \subsetneqq X$ then $x_i \in Y$. In the opposite case it would be $x_i \in \mathscr{L}(X - \{x_i\})$ and then $x_1 \in \mathscr{L}(X) \subseteq$ $\subseteq \mathscr{L}(X - \{x_i\})$ which contradicts to $x_1 \notin \mathscr{L}(Y)$ for $Y \subsetneqq X$. Hence all the sets Y_0, Y_1, Y_2, Y_3, Y_4, $\mathscr{L}(Y_1 \cup Y_2)$, $\mathscr{L}(Y_1 \cup Y_3)$, $\mathscr{L}(Y_1 \cup Y_4)$, $\mathscr{L}(Y_2 \cup Y_3)$, $\mathscr{L}(Y_2 \cup Y_4)$, $\mathscr{L}(Y_3 \cup Y_4)$ are different. From condition $(**)$ it follows that all the sets $\mathscr{L}(Y_i \cup Y_j \cup Y_k)$ for $i \neq j \neq k \neq i$, $i, j, k = 1, 2, 3, 4$ are identical, and from semimodularity of L and 4.5 it follows that $Y_0 \prec Y_i \prec \mathscr{L}(Y_i \cup Y_j) \prec \mathscr{L}(Y_i \cup Y_j \cup Y_k)$ for $i \neq j \neq k \neq i$, $i, j, k = 1, 2, 3, 4$ (here $A \prec B$ denotes that B covers A in $\mathscr{L}$).

Condition DC 6 implies the existence of Y_5 $(Y_5 \succ Y_0)$ such that, for example, $Y_5 \subseteq \mathscr{L}(Y_3 \cup Y_4)$ and $Y_5 \subseteq \mathscr{L}(Y_1 \cup Y_2)$. Since L is atomic and semimodular, there exists $x_5 \in J$ such that $Y_5 =$ $= \mathscr{L}(Y_0 \cup \{x_5\})$. If we denote $X_0 = X - \{x_1, x_2, x_3, x_4\}$, we have by the induction assumption:

$$x_5 \in \mathscr{L}(X_0 \cup \{x_3, x_4\}) = \mathscr{K}(X_0 \cup \{x_3, x_4\})$$

and

$$x_1 \in \mathscr{L}(X_0 \cup \{x_2, x_5\}) = \mathscr{K}(X_0 \cup \{x_2, x_5\}),$$

hence

$$x_1 \in \mathscr{K}(X_0 \cup \{x_2, x_3, x_4\}) = \mathscr{K}(X).$$

4.8. Theorem. *A finite lattice L is isomorphic to the product of equivalence lattices iff L is semimodular and every interval in L of the length less than five is isomorphic to some product of equivalence lattices.*

Proof. Since intervals in the product of some lattices are products of intervals in these lattices, one of the two implications follows from 4.4.

Let L be a finite semimodular lattice. We shall show that L satisfies conditions of Theorem 4.7. L is atomic, since each interval of length two in L has the form Eq (3) or Eq (2) $\times$ Eq (2). Also condition DC 2 is easily seen. From semimodularity of L it follows that every subdiagram of L of the form given by Figures 2, 4, 6, 8 can be completed to an interval in L of length less than five. But these intervals are isomorphic to some product of equivalence lattices. So by 4.7, they can be completed to the corresponding diagrams on Figures 3a or 3b, 5, 7, 9a or 9b. Using the opposite implication of 4.7, we conclude that L is isomorphic to some product of equivalence lattices.

4.9. Theorem. *A lattice L is isomorphic to some product of equivalence lattices if and only if L is complete, semimodular, algebraic, each element of L is the join of some set of atoms and each interval in L of the length less than five is isomorphic to some product of equivalence lattices.*

Proof. It follows from 4.4 and from the properties of equivalence lattices proved e.g. in [4] that each lattice L isomorphic to some product of equivalence lattices has the properties stated in the theorem.

To prove the converse implication, let again J denote the set of all atoms of L. Let $\mathscr{L}$ be the closure system on J defined by

$$x \in \mathscr{L}(X) \quad \text{iff} \quad x \leqslant \bigvee X \quad \text{in} \quad L.$$

Since each element of L is a join of atoms and L is complete, $\mathscr{L}$ is isomorphic to L.

Let $(J, S, \mathscr{K})$ be the dependence system, where $S \subseteq P_3(J)$ is defined by

$$\{a, b, c\} \in S \quad \text{if} \quad a < b \vee c \quad \text{in} \quad L.$$

Since each interval in L of length less than five is isomorphic to some product of equivalence lattices, the set S satisfies conditions EC 1 - EC 4 hence $\mathscr{K}$ is isomorphic to some product of equivalence lattices.

We shall show that for every $X \subseteq J$ it is $\mathscr{L}(X) = \mathscr{K}(X)$. From the definitions of $\mathscr{L}$ and $\mathscr{K}$ we have $\mathscr{K}(X) \subseteq \mathscr{L}(X)$ for every $X \subseteq J$. If $x \in \mathscr{L}(X)$ for some $X \subseteq J$, then, since x is an atom and thereby compact, there exists a finite $Y \subseteq X$ such that $x \in \mathscr{L}(Y)$. From semimodularity of L it follows that each maximal chain contained in the interval $[0, \bigvee Y]$ (0 is the least element of L) has a finite length n.

Let us denote

$$A_k = \{x \leqslant \bigvee Y \mid \text{the interval } [0, x] \text{ has the length } k\}.$$

From the fact that each interval in L of the length two is isomorphic to Eq (3) or Eq (2) $\times$ Eq (2) we have that if A_1 is infinite, then there exists an atom $x \in A_1$ such that the set $\{y \mid x \prec y \leqslant \bigvee Y\}$ is also infinite. From semimodularity of L, by the same consideration we can conclude that if A_1 is infinite, then each A_k, $k = 1, \ldots, n$ is infinite, too. But $A_n = \{\bigvee Y\}$ is finite, thus also A_1 is finite and the whole interval $[0, \bigvee Y]$ is finite.

Now, from 4.8 it follows that $[0, \bigvee Y]$ is isomorphic to some product of equivalence lattices, therefore if $(J_1 = J \cap [0, \bigvee Y], S \cap P_3(J_1), \mathscr{K}_1)$ is a dependence system, $\mathscr{K}_1$ is isomorphic to $[0, \bigvee Y]$. Then $x \in \mathscr{L}(Y)$ implies that $x \in \mathscr{K}_1(Y) \subseteq \mathscr{K}(Y) \subseteq \mathscr{K}(X)$, thus $\mathscr{L}(X) \subseteq \mathscr{K}(X)$ for every $X \subseteq J$.

4.10. Remark. A different approach to characterization of finite equivalence lattices can be found in [1], using the relationship of copoints, while our Theorem 4.3 characterizes equivalence lattices (or their products) through relations among their atoms.

The general characterization theorem in [1] is stated for finite lattices and uses the intervals of length 4, while Theorem 4.9 characterizes products of arbitrary (both finite and infinite) equivalence lattices trough intervals of length 5. The mutual logical relationship of these two different characterization theorems is not quite clear.

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