Retracts of universal homogeneous structures

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Dedicated to the memory of my friend
Paweł Waszkiewicz

Outline

Main goal

Theorem

Let $\langle X, d\rangle$ be a separable complete metric space. TFAE:

(a) *$\langle X, d\rangle$ is a non-expansive retract of the Urysohn space $\mathbb{U}$.*

(b) *$\langle X, d\rangle$ is finitely hyperconvex, that is, given a finite family of closed balls*

$$\mathcal{F} = \{\overline{\mathrm{B}}(x_0, r_0), \ldots, \overline{\mathrm{B}}(x_{n-1}, r_{n-1})\}$$

with $\bigcap \mathcal{F} = \emptyset$, there exist $i < j < n$ such that

$$d(x_i, x_j) > r_i + r_j.$$

Remark

Implication (a) $\implies$ (b) is easy.

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Theorem

Let K be a compact space of weight $\aleph_1$. TFAE:

(a) *K is a topological retract of ω^*.*

(b) *K is a 0-dimensional F-space, that is, disjoint open F_σ sets have disjoint closures.*

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Motivation

Theorem (Dolinka 2011)

Let $\mathfrak{M}$ be a nice Fraïssé class of finite models and let U be its Fraïssé limit. Given a countable model X, TFAE:

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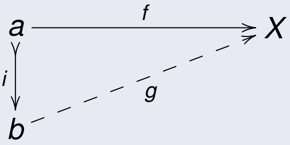
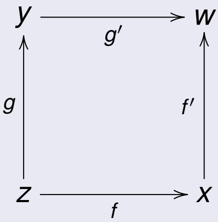
Injectivity

Definition

Let $\mathfrak{K} \subseteq \mathcal{C}$ be a pair of categories. We say that $X \in \mathrm{Ob}(\mathcal{C})$ is **$\mathfrak{K}$-injective in $\mathcal{C}$** if for every $\mathfrak{K}$-arrow $i \colon a \to b$ and for every $\mathcal{C}$-arrow $f \colon a \to X$, there exists a $\mathcal{C}$-arrow $g \colon b \to X$ such that $g \circ i = f$.



The pushout of $\langle f, g \rangle$



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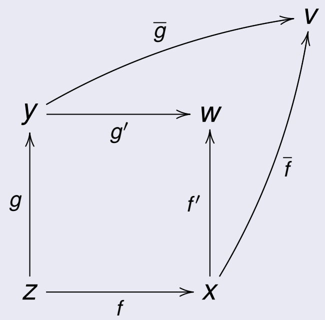

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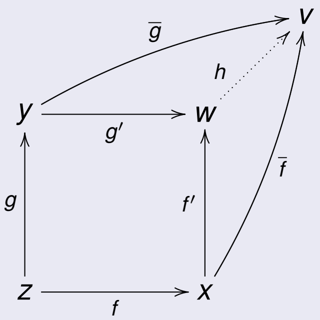
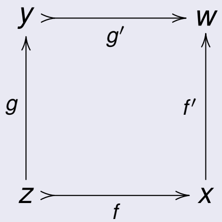


The mixed pushout property

Mixed pushout: $f, g' \in \mathfrak{K}$ and $f', g \in \mathfrak{L}$



Theorem

Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories with the same objects and satisfying the following conditions:

(h1) *$\mathfrak{K}$ has a weakly initial object.*

(h2) *$\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed pushout property.*

(h3) *$\mathfrak{K}$ has a Fraïssé sequence U.*

Let X be a sequence in $\mathfrak{K}$. The following properties are equivalent.

(a) *X is $\mathfrak{K}$-injective.*

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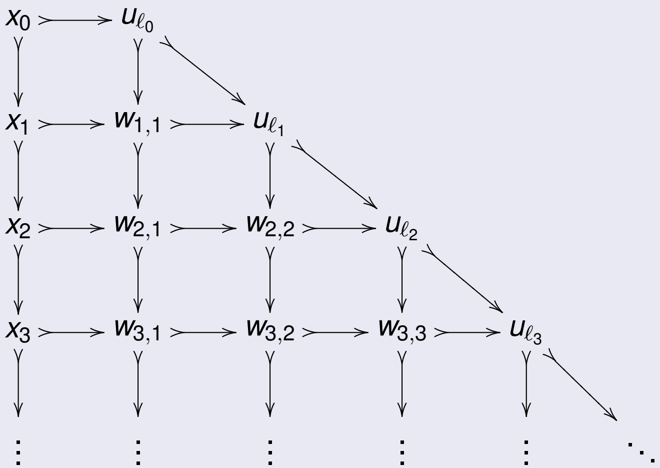
About the proof



Definition (Cameron & Nešetřil 2006)

A countable relational structure X is homomorphism homogeneous (HH) if every homomorphism between its finite substructures extends to an endomorphism of X.

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Let $\mathfrak{K} \subseteq \mathfrak{L}$ be a pair of categories with the same objects, satisfying conditions (h1) – (h3) above. Let X be a sequence in $\mathfrak{K}$. The following properties are equivalent:

(a) *X is HH.*

(b) *X is a retract of a Fraïssé sequence of some subcategory of $\mathfrak{K}$ satisfying (h1) – (h3).*

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THE END

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