Retractive linear orderings

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Some motivations

Preprint:

O. Kalenda, W. Kubiś, *The structure of Valdivia compact lines*, to appear in Topology Appl. (http://arxiv.org/abs/0811.4144)

Let $\langle X, \langle \rangle$ be a line (= linearly ordered set). Then:



For every $f : \omega \to X$ there exists an infinite set $B \subseteq \omega$ such that $f \upharpoonright B$ is monotone.

Let κ be an uncountable regular cardinal. We shall say that $\langle X, < \rangle$ has property (\mathfrak{s}_{κ}) if, given a function

 $f: S \to X$

with *S* stationary in κ , there exists a stationary set $T \subseteq S$ such that $f \upharpoonright T$ is monotone.

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For every $f: \omega \to X$ there exists an infinite set $B \subseteq \omega$ such that $f \upharpoonright B$ is monotone.

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Critical uncountable lines

$\bigcirc \omega_1$

$2 \omega_1^-$

Countryman line

- (Countryman line)⁻¹
- Incountable subset of \mathbb{R}

Theorem (J. Moore)

PFA implies that every uncountable line contains a copy of one of the above lines.

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A line is scattered if $\mathbb{Q} \not\hookrightarrow X$.

Hausdorff Theorem implies:

Proposition

Every scattered line satisfies (\mathfrak{s}_{κ}) for every regular cardinal $\kappa \geq \aleph_0$.

Lemma

Assume $\kappa = \operatorname{cf} \kappa > \aleph_0$, $S \subseteq \kappa$ is stationary, X is a set and $f : S \to X$ is a function. Then one of the following possibilities occur:

- There exists a stationary set $T_0 \subseteq S$ such that $f \upharpoonright T_0$ is constant.
- There exists a stationary set $T_1 \subseteq S$ such that $f \upharpoonright T_1$ is one-to-one.

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Kurepa's example

Define

 $Y = \{x \in 2^{\omega_1} : |\operatorname{suppt}(x)| < \aleph_0\} \cup \{1_{c_\alpha} : \alpha \in S\},\$

where $S \subseteq \omega_1$ is stationary and $\{c_\alpha\}_{\alpha \in S}$ is a ladder system. That is, $c_\alpha \approx \omega$ and $\sup c_\alpha = \alpha$ for $\alpha \in S$.

Claim

The line Y fails $(\mathfrak{s}_{\aleph_1})$, but it satisfies

 $(\forall f \colon \omega_1 \to Y) (\exists T \in [\omega_1]^{\omega_1}) f \upharpoonright T$ is monotone.

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Definition

A line X is retractive if, given a big enough cardinal θ , for every countable $M \leq \langle H(\theta), \in \rangle$ such that $X \in M$, there exists an increasing map $r: X \to X \cap M$ satisfying

$$f \upharpoonright (X \cap M) = \mathrm{id}_{X \cap M}.$$

Claim

A line X is retractive iff

$$X=\bigcup \mathcal{F},$$

where \mathcal{F} is a family of countable subsets of X satisfying

- **1** For every $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $A \cup B \subseteq C$.
- ② $\bigcup_{n \in \omega} A_n \in \mathcal{F}$ whenever $A_0 \subseteq A_1 \subseteq ...$ is a sequence in \mathcal{F} .
- ③ For every $F \in F$ the inclusion $F \subseteq X$ is left-invertible, i.e. there is an increasing map $r: X \to F$ such that $r \upharpoonright F = id_F$.

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Let

$$\mathbb{U}_{\kappa} = \{ x \in \mathbb{Q}^{\kappa} \colon |\operatorname{suppt}(x)| < \aleph_0 \}.$$

Claim

 \mathbb{U}_{κ} is retractive.

Proof.

• Fix $M \leq H(\theta)$ so that $\kappa \in M$.

(2) Let $S = \kappa \cap M$. Then

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3 Define $r(x) = x \cdot \chi_S$ for $x \in X$.

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Every retractive line satisfies $(\mathfrak{s}_{\aleph_1})$ *.*

Proof.

- Let *X* be a retractive line and fix $y: S \to X$ with $S \subseteq \lim(\omega_1)$ stationary.
- ② Fix a continuous chain $\{M_{\alpha}\}_{\alpha < \omega_1}$ of countable elementary substructures of a suitable *H*(*θ*), so that *y* ∈ *M*₀.
- Let r_{α} : $X \to X \cap M_{\alpha}$ be an increasing retraction.
- If $\alpha \in S$ then $X \cap M_{\alpha} = \bigcup_{\xi < \alpha} (X \cap M_{\xi})$.
- $r_{\alpha}(y_{\alpha}) \in X \cap M_{\xi(\alpha)}$ with $\xi(\alpha) < \alpha$, whenever $\alpha \in S$.
- **•** Fodor's pressing-down lemma $\implies \exists T \subseteq S$ stationary, such that $r_{\alpha}(y_{\alpha}) = v$ for $\alpha \in T$.
- Let $T^+ = \{ \alpha \in T : y_\alpha \ge v \}$ and $T^- = T \setminus T^+$.

Then $\{y_{\alpha}\}_{\alpha\in T^+}$ is decreasing and $\{y_{\alpha}\}_{\alpha\in T^-}$ is increasing.

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Corollary

For every κ , \mathbb{U}_{κ} satisfies $(\mathfrak{s}_{\aleph_1})$.

Theorem (W.K. 2006)

Every retractive line of size $\leqslant leph_1$ embeds into \mathbb{U}_{leph_1} .

W.Kubiś (http://www.math.cas.cz/~kubis/)

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Every retractive line of size $\leq \aleph_1$ embeds into \mathbb{U}_{\aleph_1} .

Exact completions

A line X is exactly κ -complete if every monotone κ -sequence has a limit in X.

Theorem

Every retractive line is exactly \aleph_1 *-complete.*

Proof.

- Let *X* be a retractive line and fix a strictly increasing sequence $y: \omega_1 \to X$.
- ② Fix a countable $M \leq H(\theta)$ so that *y* ∈ *M*. Let $\delta = \omega_1 \cap M$.
- Let $r: X \to X \cap M$ be an increasing retraction. We claim that $r(y_{\delta}) = \sup_{\xi < \omega_1} y_{\xi}$.
- Clearly $r(y_{\delta}) \in M$, therefore $y_{\xi} \leq r(y_{\delta})$ for every ξ .
- Suppose $r(y_{\delta}) \neq \sup_{\xi < \omega_1} y_{\xi}$ and fix $x \in X \cap M$ with $y_{\xi} < x < r(y_{\delta})$ for every $\xi < \omega_1$. Then $r(y_{\delta}) \leq r(x) = x$, because *r* is increasing.

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A line X is exactly κ -complete if every monotone κ -sequence has a limit in X.

Theorem

Every retractive line is exactly \aleph_1 -complete.

Proof.

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Claim

Let $\kappa \ge \aleph_0$ be a regular cardinal. For every line X there exists a minimal exactly κ -complete line $c_{\kappa}(X) \supseteq X$.

 $c_{\kappa}(X)$ is the exact κ -completion of X.

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Let $\kappa > \aleph_0$ be a regular cardinal and assume X is a line satisfying (\mathfrak{s}_{κ}) . Then the exact κ -completion of X satisfies (\mathfrak{s}_{κ}) .

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• Fix $y: S \to c_{\kappa}(X)$, where $S \subseteq \kappa$ is stationary. We may assume that each y_{α} is the limit of an increasing ω_1 -sequence from X.

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- If {x_α}_{α∈T} is decreasing then so is y ↾ T. Assume {x_α}_{α∈T} is increasing.
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(b) The sequence $\{y_{\alpha}\}_{\alpha \in T_1}$ is decreasing.

Suppose T₀ = T \ T₁ is stationary and define φ(ξ) ∈ T so that y_ξ < x_{φ(ξ)} for ξ ∈ T₀. Let C ∈ club(κ) be φ-closed.

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Theorem (O. Kalenda & W.K.)

Assume X is an exactly \aleph_1 -complete line satisfying $(\mathfrak{s}_{\aleph_1})$. If $|X| \leq \aleph_1$ then X is retractive.

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- Suppose there is a stationary S ⊆ ω₁ such that X_α ⊆ X is not left-invertible for α ∈ S. Choose y_α ∈ X \ X_α that fills a gap in X_α.
- ③ $(\mathfrak{s}_{\aleph_1}) \implies \exists T \subseteq S$ stationary, such that $\{y_\alpha\}_{\alpha \in T}$ is, say, increasing. Let $y = \sup_{\alpha \in T} y_\alpha$.
- Fix a countable $M \leq H(\theta)$ so that $\{y_{\alpha}\}_{\alpha \in T}, \{X_{\alpha}\}_{\alpha \in \omega_1} \in M$ and $\delta = \omega_1 \cap M \in S$. Note that $X \cap M = X_{\delta}$ and $y \in M$.
- If *x* ∈ *X*^{δ} and *x* < *y* then, by elementarity, *x* < *y*^{ξ} for some $\xi \in T \cap \delta$; in particular *x* < *y*^{δ}.
- **(** Recall that y_{δ} should fill the gap $\langle (\leftarrow, y_{\delta}) \cap X_{\delta}, (y_{\delta}, \rightarrow) \cap X_{\delta} \rangle$.

But this is not a gap, because $y = \inf(y_{\delta}, \rightarrow) \cap X_{\delta}$.

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 $Y = \{x \in 2^{\omega_1} \colon |\operatorname{suppt}(x)| < \aleph_0\} \cup \{\mathbf{1}_{c_\alpha} \colon \alpha \in S\}.$

Claim

Y is exactly \aleph_1 -complete.

Corollary

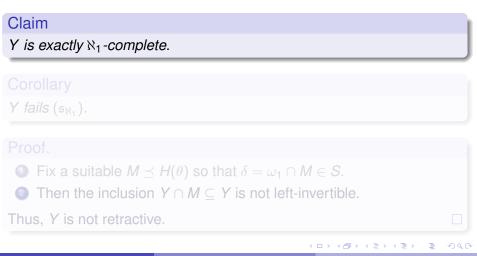
Y fails $(\mathfrak{s}_{\aleph_1})$.

Proof.

- **1** Fix a suitable $M \leq H(\theta)$ so that $\delta = \omega_1 \cap M \in S$.
- 2 Then the inclusion $Y \cap M \subseteq Y$ is not left-invertible.

Thus, Y is not retractive.

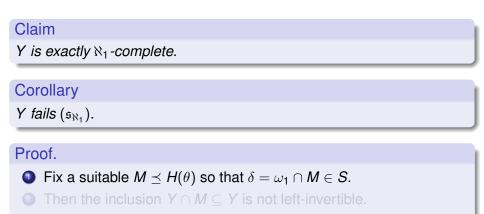
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Theorem

Assume there no Kurepa trees. Given a line X of size \aleph_1 , the following properties are equivalent:

- X satisfies $(\mathfrak{s}_{\aleph_1})$.
- 2 X embeds into a retractive line.
- \bigcirc X embeds into \mathbb{U}_{\aleph_1} .

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Natural forcing introducing a Kurepa tree

Let A be an uncountable set.

Define $\mathbb{T}(A)$ to be the set of all pairs $\langle T, f \rangle$, where

- ① $T \subseteq 2^{\leq \alpha}$ is a countable tree, $\alpha < \omega_1$,
- (a) every $t \in T$ is below some $s \in T \cap 2^{\alpha}$,
- $f: \operatorname{dom}(f) \to T \cap 2^{\alpha}$ is one-to-one and $\operatorname{dom}(f) \subseteq A$ is countable.

$$\langle T, f \rangle \leqslant \langle T', f' \rangle \Longleftrightarrow$$

• T' is an end-extension of T and • $f'(\alpha) \supseteq f(\alpha)$ for $\alpha \in \text{dom}(f)$.

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Announcement:

25th Summer Conference on Topology and its Applications will be held in Kielce (POLAND), 25 – 30 July 2010.

http://www.ujk.edu.pl/~topoconf/

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