# Banach spaces with projectional skeletons, II 

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## Plichko spaces

- A Banach space $X$ is Plichko if there are a linearly dense set $G \subseteq X$ and a norming space $D \subseteq X^{*}$ such that

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\left|\left\{x \in G:\left\langle x, x^{*}\right\rangle \neq 0\right\}\right| \leqslant \aleph_{0}
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for every $x^{*} \in D$.

- $\langle X, D\rangle$ will be called a Plichko pair.
- $X$ is weakly Lindelöf determined (WLD) if $\left\langle X, X^{*}\right\rangle$ is a Plichko pair.
- $X$ is weakly compactly generated (WCG) if $X=\mathrm{cl}$ lin $K$ for some weakly compact set K.


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Let $X$ be a Banach space with a projectional skeleton. Then every separable subspace is contained in a complemented separable space.

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## Projectional resolutions of the identity

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Let X be a Banach space with a 1-projectional skeleton {P P
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Recall that a PRI on a Banach space $X$ is a sequence of projections
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(1) $\left\|P_{\alpha}\right\|=1, P_{\kappa}=\mathrm{id} X$ and $P_{\alpha} X$ has density $\leqslant \kappa+\aleph_{0}$,
(2) $\alpha<\beta \Longrightarrow P_{\alpha} P_{\beta}=P_{\beta} P_{\alpha}=P_{\alpha}$,
(3) $P_{\delta} X=\mathrm{cl}\left(\bigcup_{\xi<\delta} P_{\xi} X\right)$ whenever $\delta \leqslant \kappa$ is a limit ordinal.

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## Retractional skeletons

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Let \(K\) be a compact, let \(\Gamma\) be an up-directed poset. An r-skeleton in \(K\)
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    (1) \(s \leqslant t \Longrightarrow r_{s} \circ r_{t}=r_{t} \circ r_{s}=r_{s}\);
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## Example

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Claim
Given an ordinal }\lambda\mathrm{ , the compact space }\lambda+1\mathrm{ has an r-skeleton.
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## Theorem (O. Kalenda 2002)

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## Elementary substructures

## Reflection Principle

Let $\varphi\left(x_{1} \ldots, x_{n}\right)$ be a formula and let $a_{1}, \ldots, a_{n}$ be fixed sets such that $\varphi\left(a_{1}, \ldots, a_{n}\right)$ is true. Then there exists a regular cardinal $\chi$ such that

$$
\langle H(\chi), \in\rangle \models \varphi\left(a_{1}, \ldots, a_{n}\right) .
$$

Löwenheim-Skolem Theorem
Assume $A \subseteq H(\chi)$. Then there exists $M \subseteq H(\chi)$ such that $A \subseteq M$, $|M|=|A|+\aleph_{0}$ and $\langle M, \in\rangle \preceq\langle H(\chi), \in\rangle$, i.e.

$$
\begin{aligned}
& \forall \varphi\left(x_{1}, \ldots, x_{n}\right) \forall a_{1}, \ldots, a_{n} \in M, \\
& M \models \varphi\left(a_{1}, \ldots, a_{n}\right) \Longleftrightarrow H(\chi) \models \varphi\left(a_{1}, \ldots, a_{n}\right) .
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## Elementary substructures

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Let $\varphi\left(x_{1}, \ldots, x_{n}\right)$ be a formula and let $a_{1}, \ldots, a_{n}$ be fixed sets such that $\varphi\left(a_{1}, \ldots, a_{n}\right)$ is true. Then there exists a regular cardinal $\chi$ such that

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\langle H(\chi), \in\rangle \models \varphi\left(a_{1}, \ldots, a_{n}\right) .
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## Löwenheim-Skolem Theorem

Assume $A \subseteq H(\chi)$. Then there exists $M \subseteq H(\chi)$ such that $A \subseteq M$, $|M|=|A|+\aleph_{0}$ and $\langle M, \epsilon\rangle \preceq\langle H(\chi), \in\rangle$, i.e.

$$
\begin{aligned}
& \forall \varphi\left(x_{1}, \ldots, x_{n}\right) \forall a_{1}, \ldots, a_{n} \in M, \\
& M \models\left(a_{1}, \ldots, a_{n}\right) \Longleftrightarrow H(\chi) \models \varphi\left(a_{1}, \ldots, a_{n}\right) .
\end{aligned}
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## Lemma

Iet $X$ be a Banach space, let $D \subseteq X^{*}$ be r-norming and let $\chi$ be a big enough regular cardinal. Further, let $M \preceq H(\chi)$ be such that $D \in M$. Then

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\operatorname{cl}(X \cap M) \cap \perp(D \cap M)=\{0\}
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Moreover, the canonical projection
$P_{M}: \operatorname{cl}(X \cap M) \oplus{ }^{\perp}(D \cap M) \rightarrow \operatorname{cl}(X \cap M)$ has norm $\leqslant r$.

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## Let $D \subseteq X$ be a norming set. We say that $\langle X, D\rangle$ generates projections

 if for every sufficiently closed countable $M \preceq H(\chi)$ we have that$$
X=\operatorname{cl}(X \cap M) \oplus^{\perp}(D \cap M)
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## Theorem

Let $X$ be a Banach space and let $D \subseteq X^{*}$ be a norming set. TFAE:

- $\langle X, D\rangle$ generates projections.
(2) There exists a projectional skeleton $\left\{P_{s}\right\}_{s \in \Gamma}$ in $X$ such that

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D \subseteq \bigcup_{S \in \Gamma} P_{S}^{*} X^{*} .
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Theorem (I. Bandlow 1994)
Let K be a comnact. The following are properties are equivalent:
(1) \(K\) is Corson compact.
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## Claim

$\langle X, D\rangle$ has Property $\Omega \Longleftrightarrow\langle X, D\rangle$ generates projections.

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A Banach space X is weakly Lindelöf determined if and only if {X, X*)
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## Theorem Let $X$ be a W/CG space and let $M \_H(\chi)$ be sufficiently closed. Then

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- Suppose $\varphi \in X^{*} \backslash\{0\}$ is such that $(X \cap M) \subseteq \operatorname{ker}(\varphi)$ and ${ }^{\perp}\left(X^{*} \cap M\right) \subseteq \operatorname{ker}(\varphi)$.
(2) $\varphi \in \mathrm{cl}_{*}\left(X^{*} \cap M\right)$.
(3) Fix a weakly compact $K$ which generates $X$.
(C) Suppose $p \in K$ is such that $|\varphi(p)|>\varepsilon>0$.
(5) For each $x \in \mathrm{cl}(K \cap M)$ choose $\psi_{x} \in X^{*} \cap M$ so that $\left|\psi_{x}(x)\right|<\varepsilon$ and $\left|\psi_{x}(p)\right|>\varepsilon$.
(6) By compactness, there are $x_{1}, \ldots, x_{n} \in \mathrm{cl}(K \cap M)$ such that

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\forall x \in \mathrm{cl}(K \cap M) \exists i \leqslant n\left|\psi_{x_{i}}(x)\right|<\varepsilon .
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( Hence $M \models \forall x \in K \exists i \leqslant n\left|\psi_{x_{i}}(x)\right|<\varepsilon$.
(B) This contradicts the elementarity of $M$.

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