Banach spaces with projectional skeletons, II

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• A Banach space X is Plichko if there are a linearly dense set $G \subseteq X$ and a norming space $D \subseteq X^*$ such that

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|\{x \in G \colon \langle x, x^* \rangle \neq 0\}| \leqslant \aleph_0
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for every x^* \in D.
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• $\langle X, D \rangle$ will be called a Plichko pair.

- X is weakly Lindelöf determined (WLD) if $\langle X, X^* \rangle$ is a Plichko pair.
- X is weakly compactly generated (WCG) if X = cl lin K for some weakly compact set K.



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Let *X* be a Banach space. A projectional skeleton in *X* is a family $\{P_s\}_{s\in\Gamma}$ of projections on *X* such that

- Γ is an up-directed partially ordered set.
- ② $P_s X$ is separable for every $s \in \Gamma$.
- $X = \bigcup_{s \in \Gamma} P_s X.$
- ③ If $s_0 < s_1 < s_2 < ...$ then $t = \sup_{n \in \omega} s_n$ exists in Γ and $P_t X = \operatorname{cl}(\bigcup_{n \in \omega} P_{s_n} X)$.



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Let $\{P_s\}_{s\in\Gamma}$ be a projectional skeleton in *X*. Then there exists a closed and cofinal subset Π of Γ such that

 $\sup_{s\in\Pi}\|P_s\|<+\infty.$

Given a projectional skeleton $\mathfrak{s} = \{P_s\}_{s \in \Gamma}$, we shall always assume that $\|\mathfrak{s}\| := \sup_{s \in \Gamma} \|P_s\| < +\infty$.



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Let X be a Banach space with a projectional skeleton. Then every separable subspace is contained in a complemented separable space.

Lemma

Let $\mathfrak{s} = \{P_s\}_{s \in \Gamma}$ be a projectional skeleton in X and let $S \subseteq \Gamma$ be an up-directed set. Then the formula

$$P_S x = \lim_{s \in S} P_s x$$
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Theorem

Let *X* be a Banach space with a 1-projectional skeleton $\{P_s\}_{s\in\Gamma}$. Then *X* has a projectional resolution of the identity $\{P_\alpha\}_{\alpha\leqslant\kappa}$ such that $P_\alpha = P_{S_\alpha}$ for some up-directed set $S_\alpha \subseteq \Gamma$ ($\alpha \leqslant \kappa$).

Recall that a **PRI** on a Banach space *X* is a sequence of projections $\{P_{\alpha}\}_{\alpha \leq \kappa}$, where $\kappa = \text{dens } X$ and

■ $P_{\delta}X = cl(\bigcup_{\xi < \delta} P_{\xi}X)$ whenever $\delta \leq \kappa$ is a limit ordinal.



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Let $\{P_{\alpha}\}_{\alpha < \kappa}$ be a projectional sequence in a Banach space X and let $D \subseteq X^*$ be a norming space such that

$${\sf D} = igcup_{lpha < \kappa} {\sf P}^*_lpha {\sf D}$$

and $\langle P_{\alpha}X, P_{\alpha}^*D\rangle$ is a Plichko pair for each $\alpha < \kappa$. Then $\langle X, D\rangle$ is a Plichko pair.

Corollary

Let X be a Banach space. The following properties are equivalent.

- (a) *X* has a commutative projectional skeleton.
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each r_s[K] is metrizable;

③ if $s_0 < s_1 < ...$ in Γ then $t = \sup_{n ∈ ω} s_n$ exists and

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Let *K* be a compact, let Γ be an up-directed poset. An r-skeleton in *K* is a family of retractions $\{r_s\}_{s\in\Gamma}$ satisfying

- 2 each $r_s[K]$ is metrizable;
- **③** if $s_0 < s_1 < \ldots$ in Γ then $t = \sup_{n \in \omega} s_n$ exists and

$$r_t(x) = \lim_{n \to \infty} r_{s_n}(x)$$

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Given an ordinal λ , the compact space $\lambda + 1$ has an r-skeleton.

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Elementary substructures

Reflection Principle

Let $\varphi(x_1, \ldots, x_n)$ be a formula and let a_1, \ldots, a_n be fixed sets such that $\varphi(a_1, \ldots, a_n)$ is true. Then there exists a regular cardinal χ such that

 $\langle H(\chi),\in\rangle\models\varphi(a_1,\ldots,a_n).$

Löwenheim-Skolem Theorem

Assume $A \subseteq H(\chi)$. Then there exists $M \subseteq H(\chi)$ such that $A \subseteq M$, $|M| = |A| + \aleph_0$ and $\langle M, \in \rangle \preceq \langle H(\chi), \in \rangle$, i.e.

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 $X = \operatorname{cl}(X \cap M) \oplus {}^{\perp}(D \cap M).$

Theorem

Let X be a Banach space and let $D \subseteq X^*$ be a norming set. TFAE:

- () $\langle X, D \rangle$ generates projections.
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Let K be a compact. The following are properties are equivalent:

- K is Corson compact.
- **2** $C_p(K)$ has Property Ω .

Property Ω:

 $C_p(K)$ has Property Ω if for every sufficiently closed countable $M \leq H(\chi)$ it holds that

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$$orall f \in X \exists g \in \mathsf{cl}(X \cap M), \quad f - g \in {}^{\perp}(D \cap M).$$

Claim

 $\langle X, D \rangle$ has Property $\Omega \iff \langle X, D \rangle$ generates projections.

Corollary

A Banach space X is weakly Lindelöf determined if and only if $\langle X, X^* \rangle$ has Property Ω .



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Theorem

Let X be a WCG space and let $M \leq H(\chi)$ be sufficiently closed. Then

 $X = \operatorname{cl}(X \cap M) \oplus {}^{\perp}(X^* \cap M).$



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- Suppose $\varphi \in X^* \setminus \{0\}$ is such that $(X \cap M) \subseteq \ker(\varphi)$ and $^{\perp}(X^* \cap M) \subseteq \ker(\varphi)$.
- $@ \varphi \in \mathsf{cl}_*(X^* \cap M).$
- If X = X Fix a weakly compact K which generates X.
- Suppose $p \in K$ is such that $|\varphi(p)| > \varepsilon > 0$.
- Solution For each x ∈ cl(K ∩ M) choose ψ_x ∈ X^{*} ∩ M so that |ψ_x(x)| < ε and |ψ_x(p)| > ε.
- **(**) By compactness, there are $x_1, \ldots, x_n \in cl(K \cap M)$ such that

 $\forall x \in \mathsf{cl}(K \cap M) \exists i \leqslant n \ |\psi_{x_i}(x)| < \varepsilon.$

This contradicts the elementarity of *M*.

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Projectional skeletons, I

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