Cantor's back-and-forth method in category theory

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Outline

Cantor's back-and-forth method

Theorem (G. Cantor)

Let $\mathbb{Q}$ denote the set of rational numbers. Then:

- *Every countable linearly ordered set embeds into $\mathbb{Q}$.*
- *For every finite sets $A, B \subseteq \mathbb{Q}$, every order preserving injection $f \colon A \to B$ extends to an order isomorphism $F \colon \mathbb{Q} \to \mathbb{Q}$.*
- *$\mathbb{Q}$ is a unique (up to order isomorphism) countable linearly ordered set with the above properties.*

Corollary

$\mathbb{Q}$ is the unique countable dense linear order with no end-points.

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Proof.

- Let $\mathbb{Q} = \bigcup_{n\in\omega} Q_n$, where each Q_n is finite and $Q_n \subseteq Q_{n+1}$.
- Let $P = \bigcup_{n\in\omega} P_n$ be a linearly ordered set, where $P_n \subseteq P_{n+1}$ and each P_n is finite.
- Define inductively embeddings $f_n \colon P_n \to Q_{k_n}$ so that $f_{n+1} \restriction P_n = f_n$.
- Now assume $P = \mathbb{Q}$ and $f \colon A \to B$ is given, where $A, B \subseteq Q_{k_0}$.
- Extend f to $f_1 : Q_{k_0} \to Q_{k_1}$, where $k_1 > k_0$.
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□

Theorem (R. Fraïssé 1954)

Let $\mathbb{M}$ be a countable class of finitely generated models of a fixed countable first-order language, satisfying the following conditions:

- *For every $A, B \in \mathbb{M}$ there is $C \in \mathbb{M}$ such that both A and B embed into C. (Joint Embedding)*
- *For every two embeddings $f \colon E \to A$ and $g \colon E \to B$, where $E, A, B \in \mathbb{M}$, there exist $D \in \mathbb{M}$ and embeddings $f' \colon A \to D$, $g' \colon B \to D$ such that $f' \circ f = g' \circ g$. (Amalgamation)*

Then there exists a unique, up to isomorphism, countable model M of the same language such that:

- *Every $A \in \mathbb{M}$ embeds into M.*
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Theorem (P.S. Urysohn 1927)

There exists a unique complete separable metric space $\mathbb{U}$ with the following properties:

- *Every separable metric space is isometric to a subset of $\mathbb{U}$.*
- *For every finite sets $A, B \subseteq \mathbb{U}$, every isometry $f\colon A \to B$ extends to an isometric bijection $F\colon \mathbb{U} \to \mathbb{U}$.*

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Categories

Let $\mathfrak{K}$ be a category.

- We say that $\mathfrak{K}$ has the amalgamation property if for every arrows $f\colon Z \to X$ and $g\colon Z \to Y$ there are arrows $f'\colon X \to W$ and $g'\colon Y \to W$ such that $f' \circ f = g' \circ g$.

$$\begin{array}{ccc} Y & \xrightarrow{g'} & W \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ Z & \xrightarrow[f]{} & X \end{array}$$

- If moreover for every other pair of arrows $k\colon X \to V$ and $\ell\colon Y \to V$ with $k \circ f = \ell \circ g$ there exists a unique arrow $h\colon W \to V$ such that

$$f' \circ h = k \text{ and } g' \circ h = \ell$$

then $\langle f', g' \rangle$ is called the pushout of $\langle f, g \rangle$.

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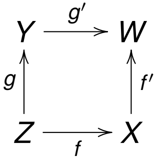

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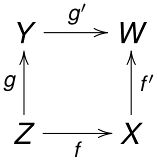

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Pushouts

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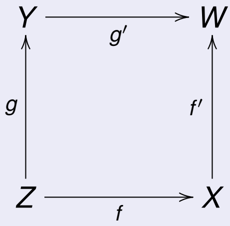

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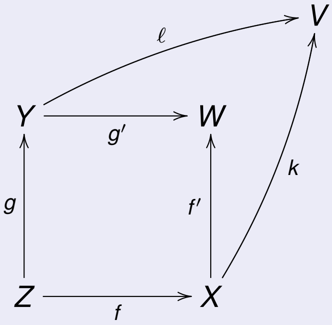

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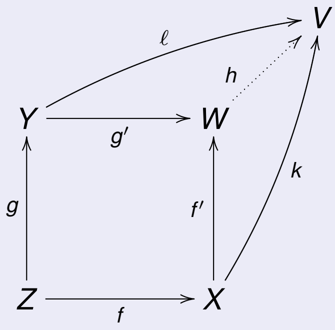

Sequences

- By a sequence in a category $\mathfrak{K}$ we mean a functor $\vec{x}$ from $\omega = \{0, 1, \dots\}$ into $\mathfrak{K}$.
- A sequence $\vec{x}$ can be described as $\{x_n\}_{n\in\omega}$ together with arrows $i_n^m \colon x_n \to x_m$ for $n \leqslant m$, such that
  1. $i_n^n = \mathrm{id}_{x_n}$,
  2. $k < \ell < m \implies i_k^m = i_\ell^m \circ i_k^\ell$.

  We shall write $\vec{x} = \langle x_n, i_n^m, \omega \rangle$.

Let $\vec{x} = \langle x_n, i_n^m, \omega \rangle$ and $\vec{y} = \langle y_n, j_n^m, \omega \rangle$ be sequences in $\mathfrak{K}$.
A transformation of $\vec{x}$ into $\vec{y}$ is a pair $\langle \varphi, \vec{f} \rangle$ such that

1. $\varphi \colon \omega \to \omega$ is increasing;
2. $\vec{f} = \{f_n\}_{n\in\omega}$, where $f_n \colon x_n \to y_{\varphi(n)}$;
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- Let $\vec{x}$, $\vec{y}$ be sequences in $\mathfrak{K}$ and let $\langle \varphi, \vec{f} \rangle$, $\langle \psi, \vec{g} \rangle$ be transformations between them. We say that they are equivalent if all diagrams like

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- An arrow of sequences $\vec{x} \to \vec{y}$ is an equivalence class of this relation.
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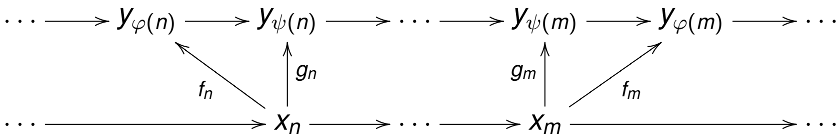

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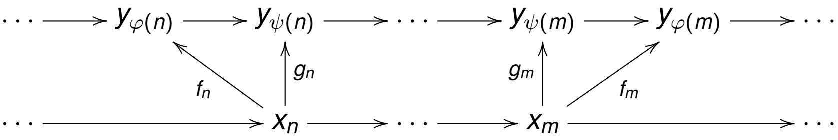

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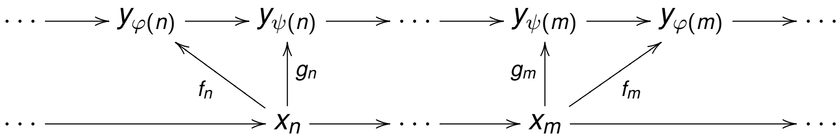

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Let $\mathfrak{K}$ be a fixed category.

A Fraïssé sequence in $\mathfrak{K}$ is a sequence $\vec{u} = \langle u_n, i_n^m, \omega \rangle$ satisfying the following conditions:

(U) For every $x \in \mathfrak{K}$ there exists $n \in \omega$ such that $\mathfrak{K}(x, u_n) \neq \emptyset$.

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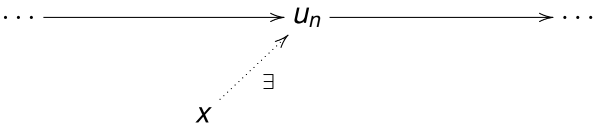

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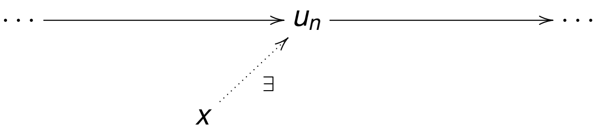

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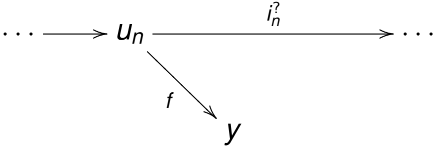

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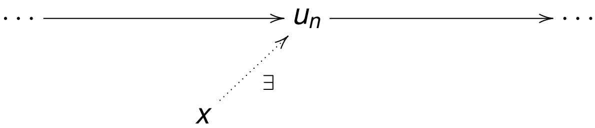

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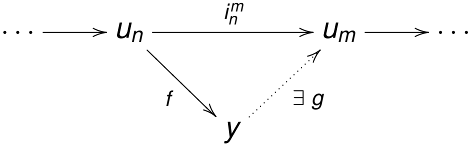

Dominating families of arrows

Let $\mathcal{F}$ be a set of arrows in $\mathfrak{K}$. Let $\mathrm{Dom}(\mathcal{F}) = \{\mathrm{dom}(f) \colon f \in \mathcal{F}\}$.

We say that $\mathcal{F}$ is dominating in $\mathfrak{K}$ if the following conditions are satisfied:

(D1) For every $x \in \mathfrak{K}$ there exists $a \in \mathrm{Dom}(\mathcal{F})$ such that $\mathfrak{K}(x, a) \neq \emptyset$.

$$x \dashrightarrow a$$

(D2) For every arrow $g \colon a \to y$ in $\mathfrak{K}$ with $a \in \mathrm{Dom}(\mathcal{F})$ there exist arrows f, h in $\mathfrak{K}$ such that $f \in \mathcal{F}$ and $f = h \circ g$.

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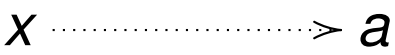

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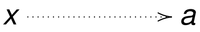

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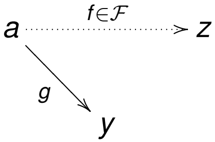

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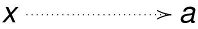

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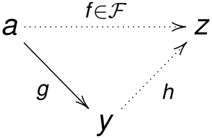

The existence

Theorem

Let $\mathfrak{K}$ be a category which has the amalgamation property and the joint embedding property. Assume further that $\mathcal{F} \subseteq \operatorname{Arr}(\mathfrak{K})$ is dominating in $\mathfrak{K}$ and $|\mathcal{F}| \leqslant \aleph_0$.
Then there exists a Fraïssé sequence $\vec{u} = \langle u_n, i_n^m, \omega \rangle$ in $\mathfrak{K}$ such that $\{u_n \colon n \in \omega\} \subseteq \operatorname{Dom}(\mathcal{F})$.

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Assume $\vec{u} = \langle u_n, i_n^m, \omega \rangle$ is a Fraïssé sequence in $\mathfrak{K}$. Then $\mathfrak{K}$ has the joint embedding property and $\mathcal{F} = \{i_n^m \colon n < m < \omega\}$ is dominating in $\mathfrak{K}$.

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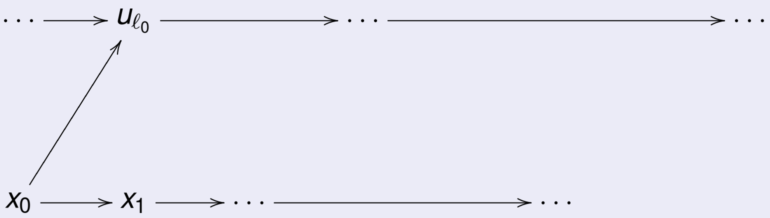

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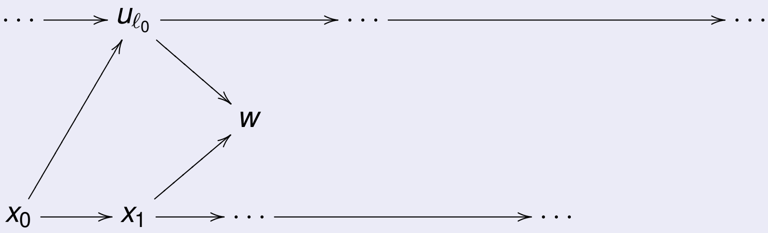

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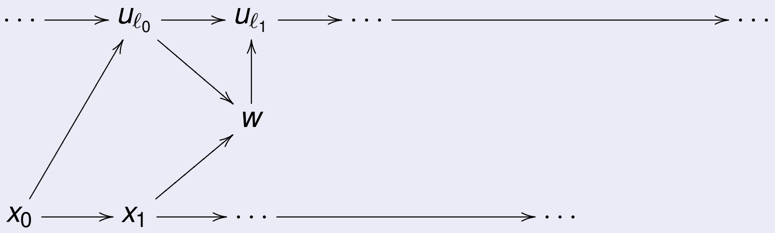

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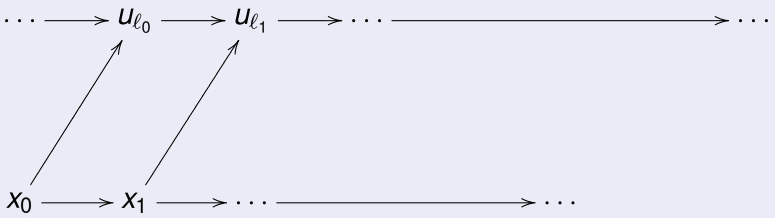

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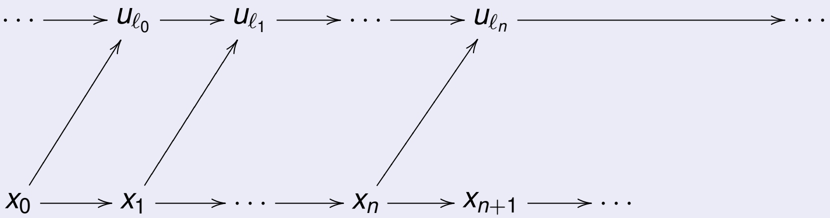

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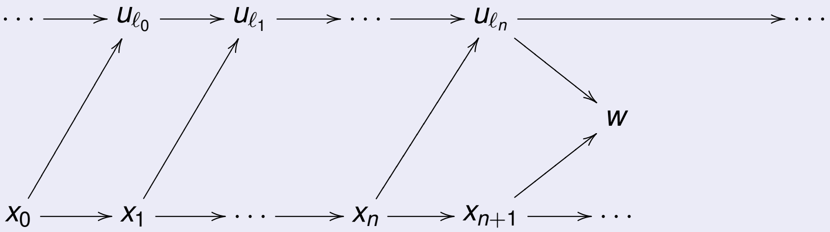

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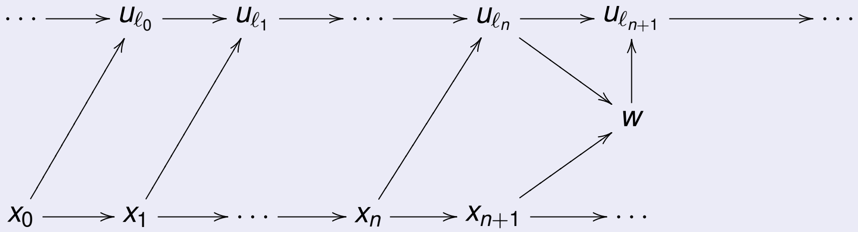

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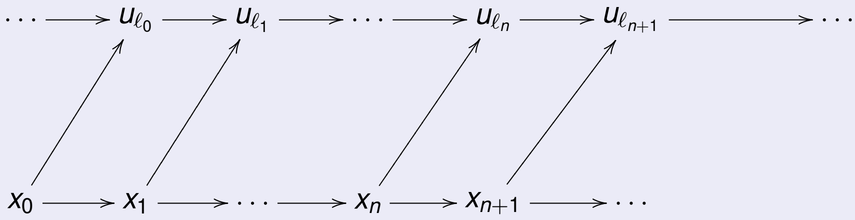
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Homogeneity and Uniqueness

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Assume that $\vec{u} = \langle u_m, i_m^n, \omega \rangle$, $\vec{v} = \langle v_m, j_m^n, \omega \rangle$ are Fraïssé sequences in a fixed category $\mathfrak{K}$.

(a) *Let $f \colon u_k \to v_\ell$, where $k, \ell < \omega$. Then there exists an isomorphism $F \colon \vec{u} \to \vec{v}$ such that $F \circ i_k = j_\ell \circ f$. In particular $\vec{u} \approx \vec{v}$.*

(b) *Assume $\mathfrak{K}$ has the amalgamation property. Then for every $a, b \in \mathfrak{K}$ and for every arrows $f \colon a \to b$, $i \colon a \to \vec{u}$, $j \colon b \to \vec{v}$ there exists an isomorphism $F \colon \vec{u} \to \vec{v}$ such that $F \circ i = j \circ f$.*

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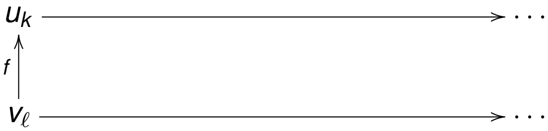

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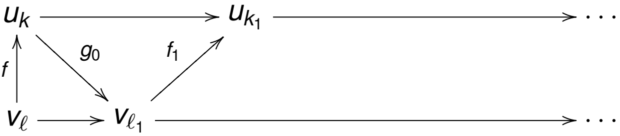

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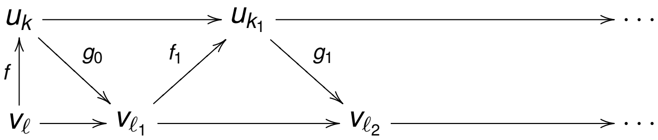

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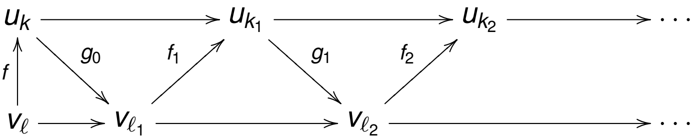

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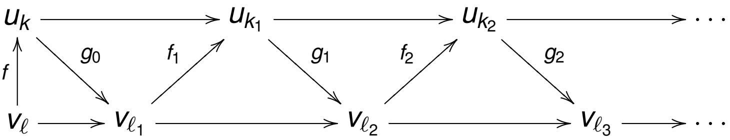

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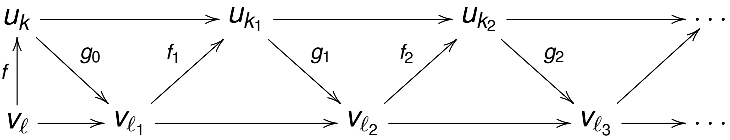

Example 1: Reversing the arrows

Let $\mathfrak{K}$ be the category described as follows:

- Objects of $\mathfrak{K}$ are finite linearly ordered sets.
- $f \in \mathfrak{K}(P, Q)$ iff $f \colon Q \to P$ is an order preserving surjection.

Claim

$\mathfrak{K}$ has the amalgamation property.

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$\mathfrak{K}$ has a Fraïssé sequence

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$$

whose limit is the Cantor set with the standard linear ordering.

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Example 2: Countable linear orders

Let $\mathfrak{K}$ be the category whose objects are countable linear orders $\langle P, \leqslant \rangle$ and arrows are left-invertible order preserving maps.

That is: $f \colon \langle P, \leqslant \rangle \to \langle Q, \preceq \rangle$ is an arrow in $\mathfrak{K}$ if

- f is order preserving, i.e. $x \leqslant y \implies f(x) \preceq f(y)$;
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Necessarily f is one-to-one.

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Example 3: Retractive pairs

Fix a category $\mathfrak{K}$. Denote by $\ddagger\mathfrak{K}$ the following category:

- The objects of $\ddagger\mathfrak{K}$ are the same as the objects of $\mathfrak{K}$.
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If $\mathfrak{K}$ has pullbacks then $\ddagger\mathfrak{K}$ has the amalgamation property.

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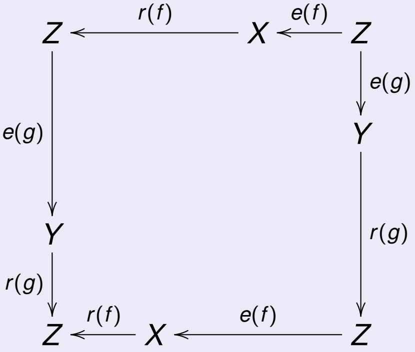

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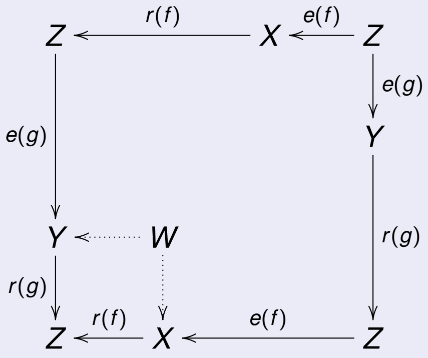

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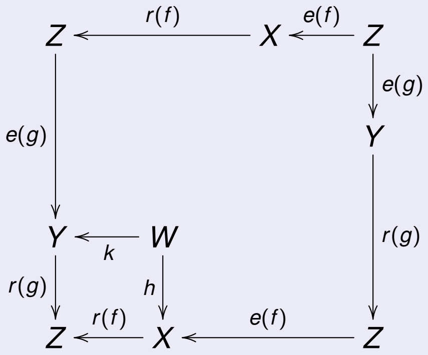

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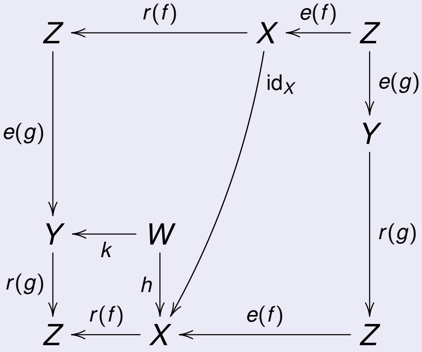

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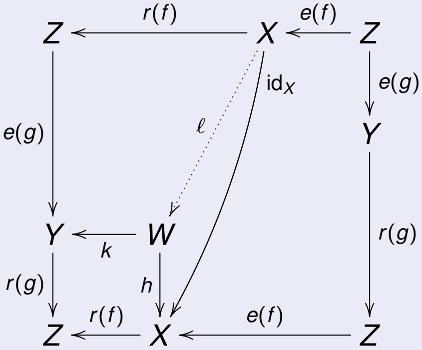

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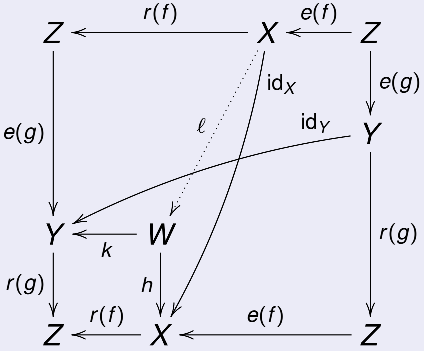

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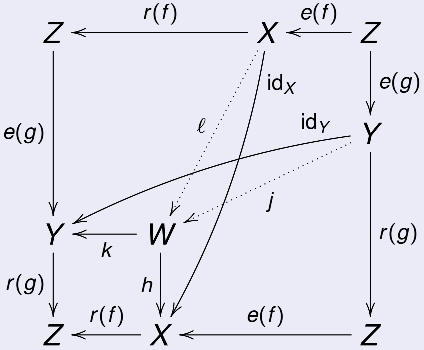

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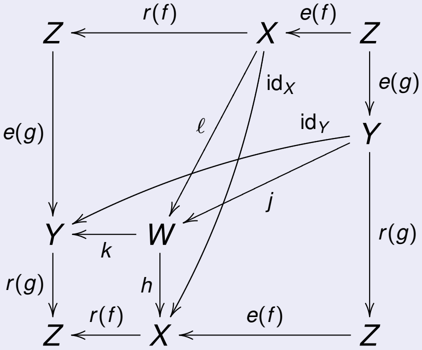

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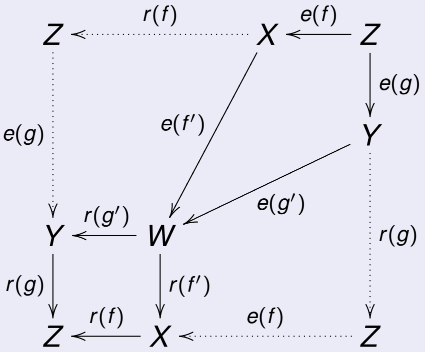

□

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