Banach spaces of universal disposition

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Gurarii space

Theorem (Gurarii, 1966)

There exists a separable Banach space $\mathbb{G}$ with the following property:

$(*)$ *Given finite-dimensional spaces $Y \subseteq X$, $\varepsilon > 0$ and an isometric embedding $i\colon Y \to \mathbb{G}$ there exists an embedding $j\colon X \to \mathbb{G}$ such that*

$$j \restriction Y = i \qquad \text{and} \qquad \max\{\|j\|, \|j^{-1}\|\} < 1 + \varepsilon.$$

A space G satisfying $(*)$ will be called of almost universal disposition for finite-dimensional spaces. Briefly: $G \in \mathrm{AUD}\,(\mathrm{fin})$.

Theorem (Lusky, 1976)

The space $\mathbb{G}$ is unique up to isometry.

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The Gurarii space is isometrically universal for separable Banach spaces.

Proposition

The space $\mathbb{G}$ *is not isomorphic to any* $\mathcal{C}(K)$.

A space G is of universal disposition for a class $\mathcal{K}$ if for every $Y \subseteq X$ with $Y, X \in \mathcal{K}$, every isometric embedding $i \colon Y \to G$ can be extended to an isometric embedding $j \colon X \to G$. We write $G \in \mathrm{UD}(\mathcal{K})$.

Theorem (Gurarii, 1966)

No separable Banach space is UD (fin).

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UD(separable)

Theorem (K., 2007)

If $2^{\aleph_0} = \aleph_1$ then there exists a unique, up to isometry, Banach space $\mathbb{U}$ of density $\aleph_1$ and of universal disposition for separable Banach spaces.

Proposition

Every Banach space of universal disposition for separable spaces is universal for spaces of density $\leq \aleph_1$.

Questions

- Does there exist a 'concrete' Banach space $U \in \mathrm{UD}\,(\mathit{separable})$?
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Digression

- The Cantor set is the unique object of universal disposition for finite sets in the category of metric compacta with quotient maps.

- Urysohn's universal metric space is of universal disposition for finite spaces + isometric embeddings.
- The space $\beta\mathbb{N} \setminus \mathbb{N}$ is UD (metric compacta + quotient maps).
- $\mathcal{C}(\beta\mathbb{N} \setminus \mathbb{N}) = \ell_\infty / c_0$.

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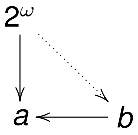

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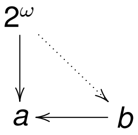

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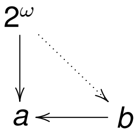

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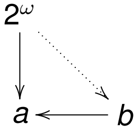

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Nonseparable Gurarii spaces

Theorem

Let X be a Banach space. Then $X \in \mathrm{AUD}\,(\mathrm{fin})$ if and only if there exists a family $\mathcal{G}$ of subspaces of X satisfying the following conditions.

- *Each $G \in \mathcal{G}$ is isometric to the Gurarii space $\mathbb{G}$.*
- *For every separable set $A \subseteq X$ there is $G \in \mathcal{G}$ with $A \subseteq G$;*

Lemma

Assume $\{X_n\}_{n\in\omega}$ is a chain such that $X = \mathrm{cl}(\bigcup_{n\in\omega} X_n)$ and each X_n is isometric to $\mathbb{G}$. Then so is X.

Corollary

No Banach space of $\mathrm{AUD}\,(\mathrm{fin})$ can be isomorphic to a $\mathcal{C}\,(K)$ space.

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Ultrapowers

Theorem (Cabello, 2008)

Let p be a non-principal ultrafilter on ω. Then $\mathbb{G}^{\omega}/_p \in \mathrm{UD}$ (separable).

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Assume $X \in \mathrm{AUD}$ (fin). Then $X^{\omega}/_p \in \mathrm{UD}$ (separable) for every non-principle ultrafilter p on ω.

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Given Banach spaces X, Y with $Z = X \cap Y$, let

$$W = (X \oplus Y)/_{\Delta_Z},$$

where $\Delta_Z = \{\langle z, -z \rangle : z \in Z\}$. Then there are isometric embeddings $i \colon X \to W$ and $j \colon Y \to W$ such that $i \restriction Z = j \restriction Z$.

Proposition

The square

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There exists a Banach space U of density $2^{\aleph_0}$ and of universal disposition for separable Banach spaces.

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A universal projection

Theorem

There exists a norm one projection $\mathfrak{u}\colon \mathbb{G} \to \mathbb{G}$ *such that* $\operatorname{Im}(\mathfrak{u}) \approx \mathbb{G}$ *and* $\ker(\mathfrak{u})$ *is infinite-dimensional. Moreover:*

- *For every finite-dimensional spaces* $X_0 \subseteq X_1$, $Y_0 \subseteq Y_1$, $Y_0 \subseteq X_0$, $Y_1 \subseteq X_1$, *for every norm one projections* $P\colon X_0 \to Y_0$, $Q\colon X_1 \to Y_1$ *such that* $Q \restriction X_0 = P$, *for every isometric embeddings* $i_0\colon Y_0 \to \mathbb{G}$, $j_0\colon X_0 \to \mathbb{G}$ *satisfying* $j_0 \restriction Y_0 = i_0$, $i_0 \circ P = \mathfrak{u} \circ j_0$, *for every* $\varepsilon > 0$ *there exist* ε*-isometric embeddings* $i\colon Y_1 \to \mathbb{G}$ *and* $j\colon X_1 \to \mathbb{G}$ *satisfying*

$$i \restriction Y_0 = i_0,\ j \restriction X_0 = j_0,\ j \restriction Y_1 = i \qquad \text{and} \qquad \mathfrak{u} \circ j = i \circ Q.$$

Corollary

There exists a Banach space of density $\aleph_1$ *which is* AUD (fin) *and which has a projectional resolution of the identity.*

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