Fraïssé sequences

Category-theoretic approach

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Outline

Motivations

- Fraïssé-Jónsson theory of universal homogeneous structures (1953)
  - Cantor's back-and-forth method
  - Urysohn's universal metric space
- Work of Droste & Göbel (1989)
- Reversed Fraïssé limits: Irwin & Solecki (2005)
- Universal compact spaces "generated" by retractions (Valdivia compacta)

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Valdivia compacta

A Valdivia compact is a closed set $K \subseteq [0,1]^\kappa$ such that

$$K = \mathrm{cl}(K \cap \Sigma(\kappa)),$$

where $\Sigma(\kappa) = \{x \in [0,1]^\kappa \colon |\{\alpha \colon x(\alpha) \neq 0\}| \leqslant \aleph_0\}$.

Theorem (H.Michalewski & W.K.)

A compact space K of weight $\aleph_1$ is Valdivia if and only if

$$K = \varprojlim \vec{s},$$

where $\vec{s} = \langle K_\xi, r_\xi^\eta, \omega_1 \rangle$ is a continuous inverse sequence of metric compacta in which all bonding maps r_ξ^η are retractions.
(retraction = right-invertible map)

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Linearly ordered Valdivia compacta

Example

The linearly ordered space $\omega_1 + 1$ is Valdivia compact.

Claim

Linearly ordered Valdivia compacta can have weight at most $\aleph_1$.

Theorem (W.K.)

There exists a zero-dimensional linearly ordered Valdivia compact C_{ω_1} which maps increasingly onto any other nonempty linearly ordered Valdivia compact.

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Categories

Let $\mathfrak{K}$ be a category. We say that $\mathfrak{K}$ has the amalgamation property if for every arrows $f\colon Z \to X$ and $g\colon Z \to Y$ there are arrows $f'\colon X \to W$ and $g'\colon Y \to W$ such that $f'f = g'g$. If moreover for every other pair of arrows $k\colon X \to V$ and $\ell\colon Y \to V$ with $kf = \ell g$ there exists a unique arrow $h\colon W \to V$ such that the diagram

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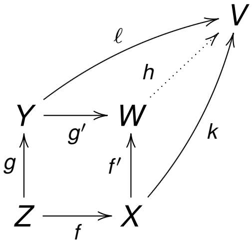

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Cofinality and homogeneity

- A family $\mathcal{F}$ of objects of $\mathfrak{K}$ is said to be cofinal in $\mathfrak{K}$ if for every $x \in \mathfrak{K}$ there is $y \in \mathcal{F}$ such that $\mathfrak{K}(x, y) \neq \emptyset$.
- An object $u \in \mathfrak{K}$ is cofinal in $\mathfrak{K}$ if for every $x \in \mathfrak{K}$ there is an arrow $f\colon x \to u$ in $\mathfrak{K}$.
- Let $\mathfrak{L}$ be a subcategory of $\mathfrak{K}$. An object $u \in \mathfrak{K}$ is $\mathfrak{L}$-homogeneous if for every arrow $f\colon a \to b$ in $\mathfrak{L}$ and for every arrows $i\colon a \to u$, $j\colon b \to u$ in $\mathfrak{K}$ there exists an isomorphism $h\colon u \to u$ such that the diagram

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Sequences

By a sequence in a category $\mathfrak{K}$ we mean a functor $\vec{x}$ from an ordinal λ into $\mathfrak{K}$. A sequence $\vec{x}$ of length λ can be described as a sequence $\{x_\alpha\}_{\alpha<\lambda}$ together with arrows $i_\alpha^\beta \colon x_\alpha \to x_\beta$ for $\alpha \leqslant \beta < \lambda$, such that

1. $i_\alpha^\alpha = \mathrm{id}_{x_\alpha}$,
2. $\alpha < \beta < \gamma \implies i_\alpha^\gamma = i_\beta^\gamma i_\alpha^\beta$.

We shall write $\vec{x} = \langle x_\alpha, i_\alpha^\beta, \lambda \rangle$.

Let $\vec{x} = \langle x_\alpha, i_\alpha^\beta, \lambda \rangle$ and $\vec{y} = \langle y_\alpha, j_\alpha^\beta, \delta \rangle$ be sequences in $\mathfrak{K}$.

A transformation of $\vec{x}$ into $\vec{y}$ is a pair $\langle \varphi, \vec{f} \rangle$ such that

1. $\varphi \colon \lambda \to \delta$ is increasing;
2. $\vec{f} = \{f_\alpha\}_{\alpha<\lambda}$, where $f_\alpha \colon x_\alpha \to y_{\varphi(\alpha)}$;
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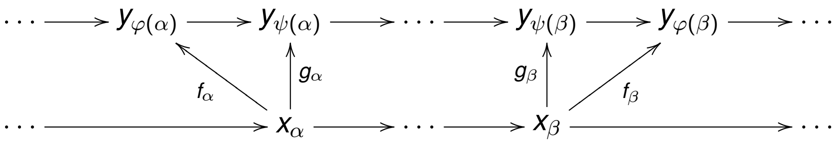
Arrows between sequences

- Let $\vec{x}$, $\vec{y}$ be sequences in $\mathfrak{K}$ and let $\langle \varphi, \vec{f} \rangle$, $\langle \psi, \vec{g} \rangle$ be transformations between them. We say that they are equivalent if all diagrams like



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- An arrow of sequences $\vec{x} \to \vec{y}$ is an equivalence class of this relation. We write $\vec{f} \colon \vec{x} \to \vec{y}$, having in mind the equivalence class of a transformation $\vec{f} = \{f_\alpha\}_{\alpha<\lambda}$.

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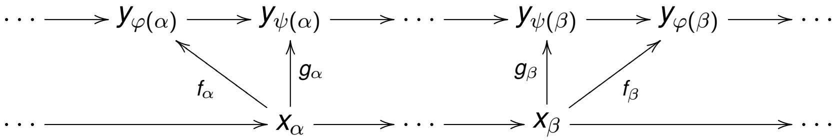

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The κ-completion of a category

Let κ be a regular cardinal and let $\mathfrak{K}$ be a category.
We denote by $\mathfrak{S}_\kappa(\mathfrak{K})$ the category of all sequences in $\mathfrak{K}$ of length $< \kappa$, with arrows of sequences defined above.
A category $\mathfrak{L}$ is κ-closed if sequences of length $< \kappa$ have colimits in $\mathfrak{L}$.

Theorem

1. *$\mathfrak{S}_\kappa(\mathfrak{K})$ is a κ-closed category containing $\mathfrak{K}$ as a full subcategory.*
2. *For every κ-closed category $\mathfrak{L}$, every covariant functor $F \colon \mathfrak{K} \to \mathfrak{L}$ has a unique extension $F' \colon \mathfrak{S}_\kappa(\mathfrak{K}) \to \mathfrak{L}$ to a κ-continuous functor.*

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If $\mathfrak{K}$ has pushouts, then $\mathfrak{S}_\kappa(\mathfrak{K})$ has the amalgamation property.

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Theorem

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Definition:

Let $\mathfrak{K}$ be a fixed category. A κ-Fraïssé sequence in $\mathfrak{K}$ is an inductive sequence $\vec{u} = \langle u_\xi, i_\xi^\eta, \kappa \rangle$ satisfying the following conditions:

(U) For every $x \in \mathfrak{K}$ there exists $\xi < \kappa$ such that $\mathfrak{K}(x, u_\xi) \neq \emptyset$.

(A) For every $\xi < \kappa$ and for every arrow $f \in \mathfrak{K}(u_\xi, y)$, where $y \in \mathfrak{K}$, there exist $\eta \geqslant \xi$ and $g \in \mathfrak{K}(y, u_\eta)$ such that $i_\xi^\eta = gf$.

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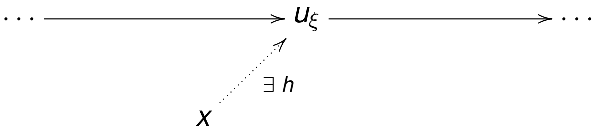

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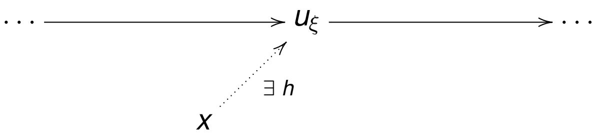

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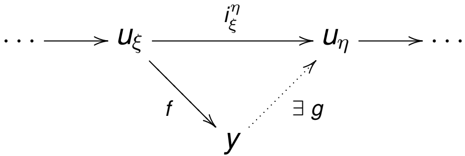

Dominating families of arrows

Let $\mathcal{F}$ be a set of arrows in $\mathfrak{K}$. Let $\mathrm{Dom}(\mathcal{F}) = \{\mathrm{dom}(f) \colon f \in \mathcal{F}\}$. We say that $\mathcal{F}$ is dominating in $\mathfrak{K}$ if the following conditions are satisfied:

(D1) For every $x \in \mathfrak{K}$ there exists $a \in \mathrm{Dom}(\mathcal{F})$ such that $\mathfrak{K}(x, a) \neq \emptyset$.

(D2) For every arrow $g \colon a \to y$ in $\mathfrak{K}$ with $a \in \mathrm{Dom}(\mathcal{F})$ there exist arrows f, h in $\mathfrak{K}$ such that $f \in \mathcal{F}$ and $f = hg$.

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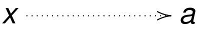

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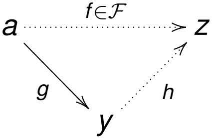

The existence

A category $\mathfrak{K}$ is κ-bounded if for every sequence $\vec{u} \in \mathfrak{S}_\kappa(\mathfrak{K})$ there are $a \in \mathfrak{K}$ and an arrow of sequences $F\colon \vec{u} \to a$.

Theorem

Let $\kappa > 1$ be a regular cardinal and let $\mathfrak{K}$ be a κ-bounded category which has the amalgamation property and the joint embedding property. Assume further that $\mathcal{F} \subseteq \mathrm{Arr}(\mathfrak{K})$ is dominating in $\mathfrak{K}$ and $|\mathcal{F}| \leqslant \kappa$.
Then there exists a Fraïssé sequence $\vec{u} = \langle u_\xi, i_\xi^\eta, \kappa \rangle$ in $\mathfrak{K}$ such that $\{u_\alpha \colon \alpha < \kappa\} \subseteq \mathrm{Dom}(\mathcal{F})$.

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Linearly ordered Valdivia compacta revisited

Denote by $\mathfrak{LV}_0$ the category whose objects are nonempty 0-dimensional metric linearly ordered compacta and arrows are increasing retractions.

Claim

The dual category $\mathfrak{LV}_0{}^{\leftarrow}$ has the amalgamation property.

Let C be the Cantor set with the standard linear order. Let $\pi \colon C \to C$ be the (unique) increasing surjection such that

- $\pi^{-1}(p)$ is order isomorphic to the Cantor set whenever $p \in C$ is rational,
- $|\pi^{-1}(p)| = 1$ whenever $p \in C$ is not rational.

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Corollary

$\mathfrak{L}\mathfrak{V}_0^{\leftarrow}$ has both an $\aleph_0$- and an $\aleph_1$-Fraïssé sequence.

In fact:

Theorem

Let $\vec{c} = \langle C_\xi, \pi_\xi^\eta, \omega_1 \rangle$ be the inverse sequence in $\mathfrak{L}\mathfrak{V}_0$ described by the following conditions:

- *$C_\xi \approx C$ and $\pi_\xi^{\xi+1} \approx \pi$ for every $\xi < \omega_1$.*
- *$\vec{c}$ is continuous with respect to the category of all compact spaces.*

Let

$$C_{\omega_1} = \varprojlim \vec{c}$$

in the category of all linearly ordered compact spaces. Then C_{ω_1} is a linearly ordered Valdivia compact which maps increasingly onto any other nonempty linearly ordered Valdivia compact.

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Countable Fraïssé sequences

Theorem (Countable Cofinality)

Assume $\vec{u} = \langle u_\alpha, i_\alpha^\beta, \kappa \rangle$ is a Fraïssé sequence in a category with amalgamation $\mathfrak{K}$. Then for every countable sequence $\vec{x}$ in $\mathfrak{K}$ there exists an arrow $\vec{f} \colon \vec{x} \to \vec{u}$.

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Let $\vec{u}$ be a countable Fraïssé sequence in a category $\mathfrak{K}$. If $\mathfrak{K}$ satisfies amalgamation then $\vec{u}$ is cofinal in $\mathfrak{S}_{\aleph_1}(\mathfrak{K})$.

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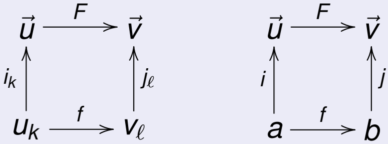
Homogeneity & Uniqueness

Theorem

Assume that $\vec{u} = \langle u_m, i_m^n, \omega \rangle$, $\vec{v} = \langle v_m, j_m^n, \omega \rangle$ *are Fraïssé sequences in a fixed category* $\mathfrak{K}$.

(a) *Let* $f\colon u_k \to v_\ell$, *where* $k, \ell < \omega$. *Then there exists an isomorphism* $F\colon \vec{u} \to \vec{v}$ *such that* $F i_k = j_\ell f$. *In particular* $\vec{u} \approx \vec{v}$.

(b) *Assume* $\mathfrak{K}$ *has the amalgamation property. Then for every* $a, b \in \mathfrak{K}$ *and for every arrows* $f\colon a \to b$, $i\colon a \to \vec{u}$, $j\colon b \to \vec{v}$ *there exists an isomorphism* $F\colon \vec{u} \to \vec{v}$ *such that* $Fi = jf$.



Uncountable Fraïssé sequences

Theorem

Let $\kappa > \aleph_0$ be regular and assume that $\mathfrak{K}$ is a full and cofinal subcategory of a κ-closed category $\mathfrak{L}$. If $\mathfrak{L}$ has the amalgamation property, then:

1. *There exists, up to equivalence of sequences, at most one κ-Fraïssé sequence in $\mathfrak{K}$.*
2. *A κ-Fraïssé sequence in $\mathfrak{K}$ is also a Fraïssé sequence in $\mathfrak{L}$ and it is both $\mathfrak{L}$-homogeneous and $\mathfrak{S}_{\kappa^+}(\mathfrak{L})$-cofinal.*

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Corollary

Let $\kappa > \aleph_0$ be regular and let $\mathfrak{K}$ be a category. Assume at least one of the following conditions is satisfied:

1. *$\mathfrak{K}$ is κ-closed.*
2. *$\mathfrak{K}$ has pushouts.*
3. *$\mathfrak{S}_\kappa(\mathfrak{K})$ has the amalgamation property.*

Then a possible κ-Fraïssé sequence in $\mathfrak{K}$ is

- *unique,*
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Trees

A tree $\langle T, \leqslant \rangle$ is bounded if for every $x \in T$ there is $y \in \max(T)$ such that $x \leqslant y$.

Let $\mathfrak{T}_2$ be the following category:

- Objects are bounded countable binary trees.
- Arrows are tree embeddings $f\colon T \to S$ such that $f[T]$ is a closed initial segment of S.

Claim

$\mathfrak{T}_2$ has the amalgamation property.

A tree T is healthy if

- every $x \in T \setminus \max(T)$ has two immediate successors,
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Let $\vec{u}$, $\vec{v}$ be $\aleph_1$-Fraïssé sequences in $\mathfrak{T}_2$, induced by healthy trees U, V respectively. Let $\vec{f}\colon \vec{u} \to \vec{v}$ be an arrow of sequences. Then the induced tree embedding $f_{\omega_1}\colon U \to V$ is an isomorphism.

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$\mathfrak{T}_2$ has at least two incomparable $\aleph_1$-Fraïssé sequences.

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Valdivia compacta

Let $\mathfrak{Vld}$ be the following category:

- The objects of $\mathfrak{Vld}$ are nonempty metric compacta.
- An arrow from $X \in \mathfrak{Vld}$ to $Y \in \mathfrak{Vld}$ is a retraction $r \colon Y \to X$.

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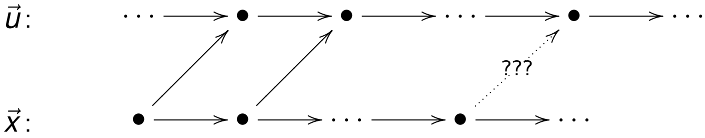

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Let $F\colon \mathfrak{Vld} \to \mathfrak{Comp}^{\leftarrow}$ be the natural functor. Then:

- Under CH, there exists an $\aleph_1$-Fraïssé sequence $\vec{u}$ in $\mathfrak{Vld}$ such that $F[\vec{u}]$ is continuous.
- Let K be a Valdivia compact of weight $\aleph_1$. Then there is a sequence $\vec{x}$ in $\mathfrak{Vld}$ such that $F[\vec{x}]$ is continuous and $K = \lim F[\vec{x}]$.

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Let $f\colon X \to Z$, $g\colon Y \to Z$ be continuous surjections between compact spaces and assume f is a retraction. Then there are a compact W and continuous surjections $f'\colon W \to X$, $g'\colon W \to Y$ such that the diagram

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- Let K be a Valdivia compact of weight $\aleph_1$. Then there is a sequence $\vec{x}$ in $\mathfrak{Vld}$ such that $F[\vec{x}]$ is continuous and $K = \lim F[\vec{x}]$.

Claim

Let $f\colon X \to Z$, $g\colon Y \to Z$ be continuous surjections between compact spaces and assume f is a retraction. Then there are a compact W and continuous surjections $f'\colon W \to X$, $g'\colon W \to Y$ such that the diagram

$$\begin{array}{ccc} Y & \xleftarrow{\;g'\;} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xleftarrow{\;f\;} & X \end{array}$$

commutes and moreover g' is a retraction.

The above claim says that:

- The functor $F\colon \mathfrak{Bld} \to \mathfrak{Comp}^{\leftarrow}$ has the amalgamation property, i.e. for every $x, y, z \in \mathfrak{Bld}$ for every arrow $f\colon z \to x$ in $\mathfrak{Bld}$ and for every arrow $g\colon F(z) \to F(y)$ in $\mathfrak{Comp}^{\leftarrow}$ there exist $w \in \mathfrak{Bld}$, an arrow $h\colon F(x) \to F(w)$ in $\mathfrak{Comp}^{\leftarrow}$ and an arrow $k\colon y \to w$ in $\mathfrak{Bld}$ such that the diagram

$$\begin{array}{ccc} F(y) & \xrightarrow{F(k)} & F(w) \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h} \\ F(z) & \xrightarrow[F(f)]{} & F(x) \end{array}$$

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Theorem

Let $\kappa > \aleph_0$ be a regular cardinal and let $\Phi \colon \mathfrak{K} \to \mathfrak{L}$ be a covariant functor with amalgamation. Assume that

- *$\vec{u}$ is a κ-Fraïssé sequence in $\mathfrak{K}$;*
- *$\Phi[\vec{u}]$ is continuous in $\mathfrak{L}$.*

Then for every sequence $\vec{x} \in \mathfrak{S}_{\kappa^+}(\mathfrak{K})$ such that $\Phi[\vec{x}]$ is continuous in $\mathfrak{L}$, there exists an arrow $\vec{f} \colon \Phi[\vec{x}] \to \Phi[\vec{u}]$ in $\mathfrak{S}_{\kappa^+}(\mathfrak{L})$.

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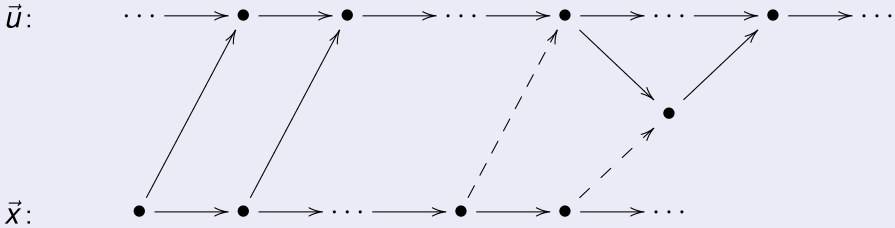

We say that a functor $\Phi\colon \mathfrak{K} \to \mathfrak{L}$ does not add isomorphisms if for every isomorphism $h\colon \Phi(a) \to \Phi(b)$ in $\mathfrak{L}$ there is an isomorphism h' in $\mathfrak{K}$ such that $h = \Phi(h')$.

Theorem

Let $\Phi\colon \mathfrak{K} \to \mathfrak{L}$ be a faithful covariant functor which does not add isomorphisms. Further, let $\vec{u}$ and $\vec{v}$ be κ-Fraïssé sequences in $\mathfrak{K}$ such that $\Phi[\vec{u}]$ and $\Phi[\vec{v}]$ are continuous in $\mathfrak{L}$. Then for every arrows $f\colon a \to b$, $i\colon a \to \vec{u}$ and $j\colon b \to \vec{v}$ in $\mathfrak{K}$ there exists an isomorphism of sequences $\vec{h}\colon \vec{u} \to \vec{v}$ in $\mathfrak{S}_{\kappa^+}(\mathfrak{K})$ for which the diagram

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Corollary

Assume CH. There exists a Valdivia compact K such that:

- *The weight of K is $\aleph_1$.*
- *Every metric compact is a retract of K.*
- *Every nonempty Valdivia compact of weight $\leqslant \aleph_1$ is a continuous image of K.*
- *For every retractions $r\colon X \to Y$, $k\colon K \to X$ and $\ell\colon K \to Y$, where X, Y are metric compacta, there exists a homeomorphism $h\colon K \to K$ such that*

$$\begin{array}{ccc} K & \xrightarrow{h} & K \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \ell} \\ X & \xrightarrow{r} & Y \end{array}$$

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Remark

If $\mathfrak{Vld}$ has an $\aleph_1$-Fraïssé sequence then CH holds.

Question

Assume CH and let K be the Valdivia compact from the above corollary. Is every Valdivia compact of weight $\aleph_1$ a retract of K?

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Retractive pairs

Fix a category $\mathfrak{K}$. Denote by $\ddagger\mathfrak{K}$ the following category:

- The objects of $\ddagger\mathfrak{K}$ are the same as the objects of $\mathfrak{K}$.
- Given $a, b \in \ddagger\mathfrak{K}$, an arrow $f \colon a \to b$ in $\ddagger\mathfrak{K}$ is a pair $f = \langle r, e \rangle$, where $r \colon b \to a$ and $e \colon a \to b$ are arrows of $\mathfrak{K}$ such that $re = \mathrm{id}_a$. We shall write $r(f) = r$, $e(f) = e$.
- Given compatible arrows f, g in $\ddagger\mathfrak{K}$, their composition is

$$gf = \langle r(f)r(g), e(g)e(f) \rangle.$$

Claim

If $\mathfrak{K}$ has pullbacks or pushouts then $\ddagger\mathfrak{K}$ has the amalgamation property.

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Proper amalgamations

Let $f\colon Z \to X$ and $g\colon Z \to Y$ be arrows in $\ddagger\mathfrak{K}$.

We say that arrows $h\colon X \to W$, $k\colon Y \to W$ provide a proper amalgamation of f, g if $hf = kg$ and moreover $i(g)r(f) = r(k)i(h)$, $i(f)r(g) = r(h)i(k)$ hold.

We say that $\ddagger\mathfrak{K}$ has proper amalgamations if every pair of arrows of $\ddagger\mathfrak{K}$ with the same domain can be properly amalgamated.

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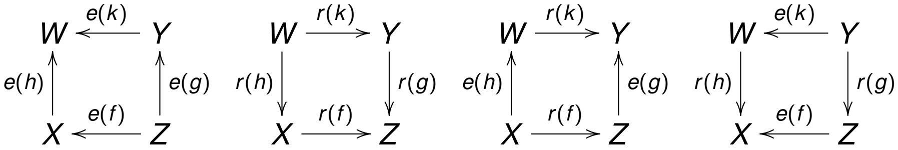

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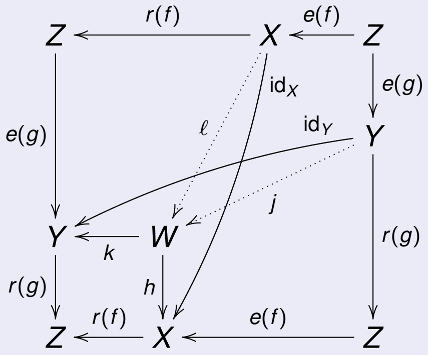

Given a category $\mathfrak{K}$, let $\Phi \colon \ddagger\mathfrak{K} \to \mathfrak{K}$ be the contravariant "forgetful" functor, i.e. $\Phi(f) = r(f)$ for every arrow f in $\ddagger\mathfrak{K}$.

We shall say that a sequence $\vec{x} \in \mathfrak{S}_\lambda(\ddagger\mathfrak{K})$ is semi-continuous if $\Phi[\vec{x}]$ is continuous.

Theorem

Let $\mathfrak{K}$ be a category such that $\ddagger\mathfrak{K}$ has proper amalgamations. Assume $\vec{u}$ is a semi-continuous κ-Fraïssé sequence in $\ddagger\mathfrak{K}$.

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Assume CH. Then there exists a Valdivia compact K of weight $\aleph_1$ such that every nonempty Valdivia compact of weight $\leqslant \aleph_1$ is a retract of K.

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Banach spaces

Let $\mathfrak{B}_{\aleph_0}$ be the category of separable Banach spaces, with arrows being linear transformations of norm $\leqslant 1$.

Claim

$\mathfrak{B}_{\aleph_0}$ has pushouts and is $\aleph_1$-closed.

Theorem

Under CH there exists a Banach space E of density $\aleph_1$ such that

- *E has a projectional resolution of the identity (PRI);*
- *every Banach space of density $\leqslant \aleph_1$ and with a PRI is linearly isometric to a one-complemented subspace of E.*

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