A general approach to universal homogeneous structures

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Outline

Some history

Theorem (Cantor 1895)

The set of rational numbers $\langle \mathbb{Q}, < \rangle$ *is the unique countable linearly ordered set whose ordering is dense and has no end-points.*

Proof.

Back-and-forth argument (developed by Huntington 1904 and Hausdorff 1914). □

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Urysohn's space

Theorem (P.S. Urysohn 1924)

There exists a unique complete separable metric space $\mathbb{U}$ satisfying the following conditions:

1. *Every separable metric space embeds isometrically into $\mathbb{U}$.*
2. *Every isometry between finite subsets of $\mathbb{U}$ extends to a bijective isometry of $\mathbb{U}$.*

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Roland Fraïssé (1920 – 2008)

A French mathematician working mostly in logic, set theory (theory of relations). One of his important works is the Fraïssé limit construction in model theory:

- ☞ *Sur quelques classifications des systèmes de relations*, Publ. Sci. Univ. Alger. Sér. A. **1** (1954) 35–182

Fraïssé theory

The setup:

A class $\mathscr{F}$ of finitely-generated models of a fixed first-order language.

Assumptions:

- If $X \in \mathscr{F}$ and Y is a submodel of X then $Y \in \mathscr{F}$.
- Given $X, Y \in \mathscr{F}$, there is $Z \in \mathscr{F}$ such that both X and Y embed into Z.
- Given $X, Y \in \mathscr{F}$ with $Z = X \cap Y \in \mathscr{F}$ there exist $W \in \mathscr{F}$ and embeddings $f\colon X \to W$, $g\colon Y \to W$ such that

$$f \restriction Z = g \restriction Z.$$

Definition

Such a family $\mathscr{F}$ is called a Fraïssé class.

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Let $\mathscr{F}$ be a class of finitely-generated models. Denote by $\overline{\mathscr{F}}$ the class of all countable models that can be written as $\bigcup_{n\in\mathbb{N}} X_n$, where $X_n \in \mathscr{F}$ for every $n \in \mathbb{N}$.

Given a countable model M, denote by $\mathrm{Age}(M)$ the class of all finitely-generated models isomorphic to submodels of M.

Theorem (Fraïssé 1954)

Let $\mathscr{F}$ be a Fraïssé class with countably many isomorphic types. Then there exists a unique model $U \in \overline{\mathscr{F}}$ satisfying the following conditions.

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Let $\mathscr{F}$ be a Fraïssé class with $U = \mathrm{Flim}(\mathscr{F})$. Then

(a) *For every isomorphism $h \colon X \to Y$ between finitely-generated submodels of U there exists an automorphism $H \colon U \to U$ such that $H \restriction X = h$.*

(b) *For every $A \in \overline{\mathscr{F}}$ there exists an embedding $f \colon A \to U$.*

Condition (a) is called ultra-homogeneity or $\mathscr{F}$-homogeneity.
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Typical examples

1. $\langle \mathbb{Q}, < \rangle = \mathrm{Flim}(\mathscr{L})$, where $\mathscr{L} =$ all finite linear orderings.
2. The random graph is $\mathrm{Flim}(\mathscr{G})$, where $\mathscr{G} =$ all finite graphs.
3. The rational Urysohn space is $\mathrm{Flim}(\mathscr{U})$, where $\mathscr{U}$ is the class of all finite metric spaces with rational distances.

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A related example

Theorem (Gurarii 1966)

There exists a separable Banach space $\mathbb{G}$ with the following property:

(G) *Given finite-dimensional spaces $X \subseteq Y$, given an isometric embedding $f\colon X \to \mathbb{G}$, given $\varepsilon > 0$, there exists an ε-isometric embedding $\bar{f}\colon Y \to \mathbb{G}$ such that $\bar{f} \restriction X = f$.*

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Uniqueness of the Gurarii space

Theorem (Solecki & K. 2011)

For every ε-isometry $f\colon X \to Y$ between finite-dimensional subspaces of separable Gurarii space G_1, G_2 respectively, there exists a bijective isometry $h\colon G_1 \to G_2$ such that

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Proof.

An "approximate" version of the back-and-forth argument, using a geometric lemma on "correcting" almost isometries. □

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  Applications to algebra and domain theory.
- 2006 T. Irwin, S. Solecki: Reversed Fraïssé theory for finite models.
  Application to topology: the pseudo-arc.

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Category-theoretic approach

Setup:

Fix a category $\mathfrak{K}$ with the following properties:

(JE) For every objects x, y in $\mathfrak{K}$ there exists a diagram of the form

$$x \longrightarrow z \longleftarrow y$$

(A) For every $\mathfrak{K}$-arrows $f\colon c \to a$, $g\colon c \to b$ there are arrows f', g' for which the diagram

$$\begin{array}{ccc} b & \overset{g'}{\dashrightarrow} & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ c & \underset{f}{\longrightarrow} & a \end{array}$$

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Fraïssé sequences

Definition

A sequence

$$u_0 \longrightarrow u_1 \longrightarrow \cdots \longrightarrow u_n \longrightarrow u_{n+1} \longrightarrow \cdots$$

is Fraïssé if the following conditions are satisfied:

(JE) For every object x in $\mathfrak{K}$ there exists an arrow $x \to u_n$ for some $n \in \mathbb{N}$.

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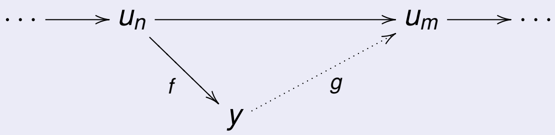

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Theorem (Existence)

Assume $\mathfrak{K}$ is dominated by a countable family of arrows. Then there exists a Fraïssé sequence in $\mathfrak{K}$.

Theorem (Homogeneity and Uniqueness)

Let $\vec{u}$, $\vec{v}$ be Fraïssé sequences in $\mathfrak{K}$ and let $f \colon u_0 \to v_0$ be a $\mathfrak{K}$-arrow. Then there exists an isomorphism of sequences $H \colon \vec{u} \to \vec{v}$ extending f.

Theorem (Universality)

Let $\vec{u}$ be a Fraïssé sequence in $\mathfrak{K}$. Then for every sequence $\vec{x}$ in $\mathfrak{K}$ there exists an arrow of sequences

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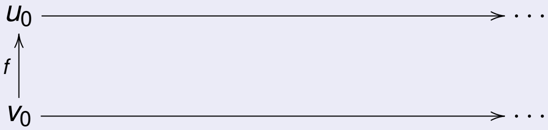

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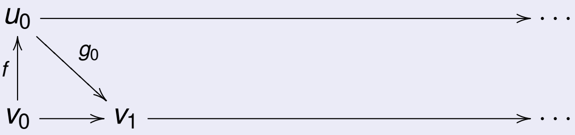

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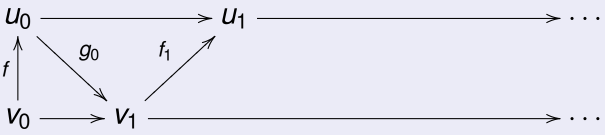

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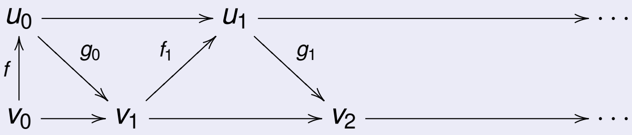

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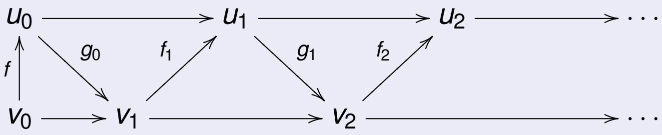

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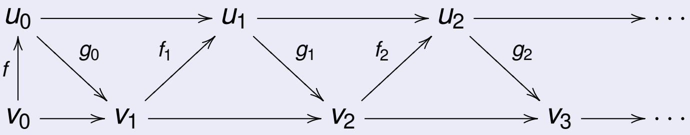

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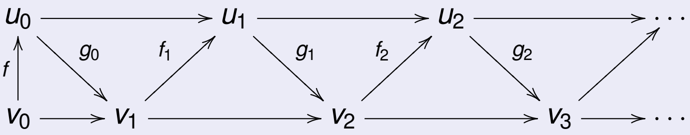

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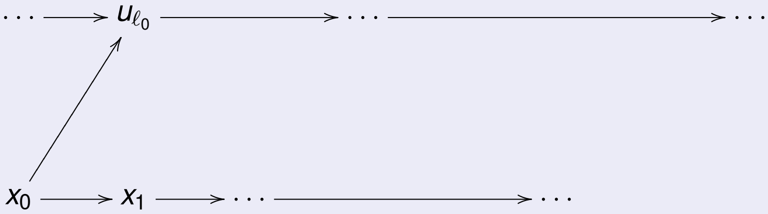

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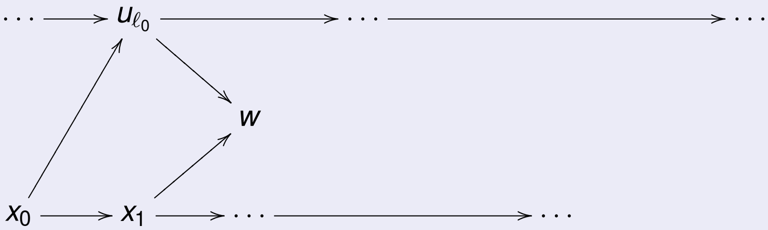

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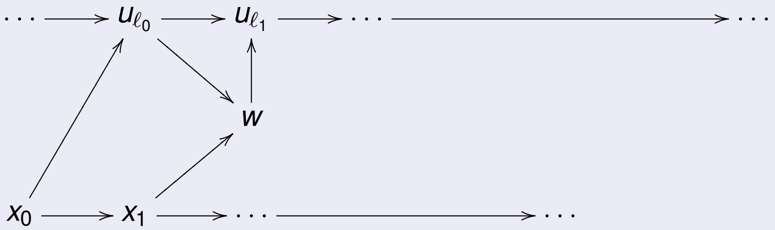

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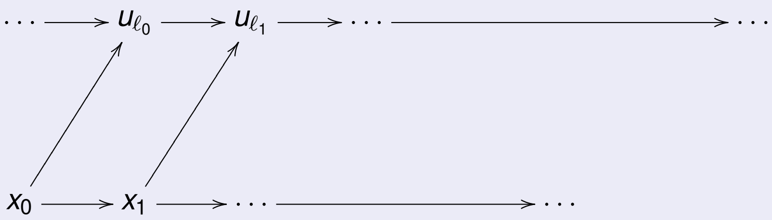

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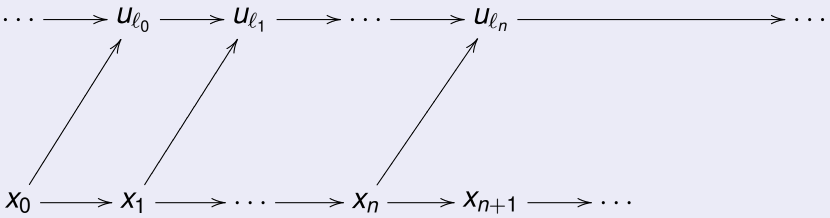

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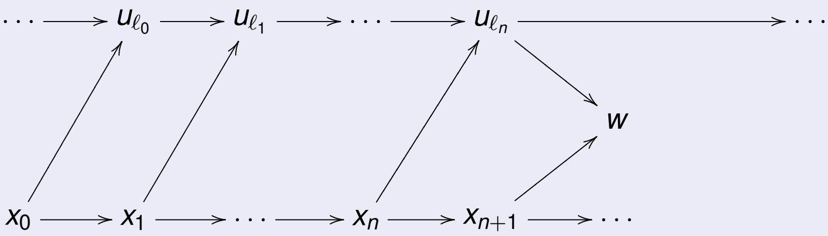

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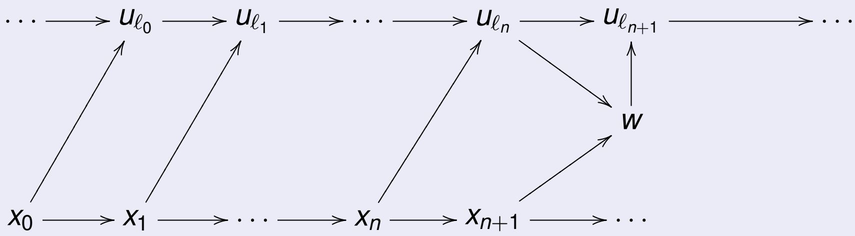

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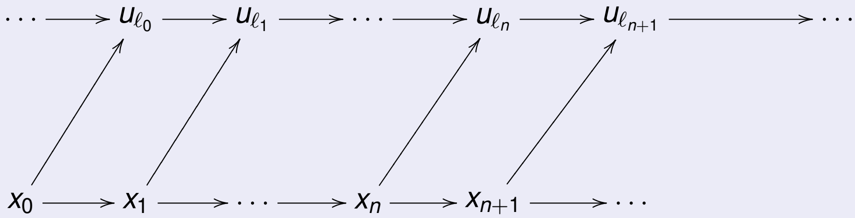

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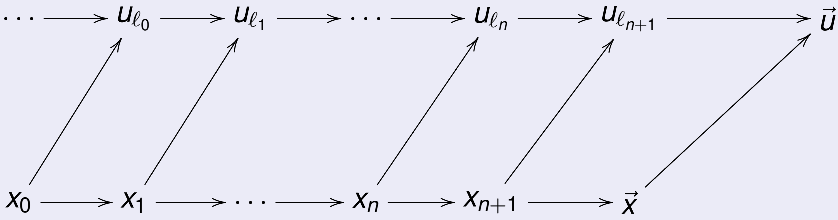

Banach spaces

Theorem

Assume the Continuum Hypothesis. There exists a unique, up to isometry, Banach space V_{ω_1} of density $\aleph_1$ with the following properties.

1. *V_{ω_1} contains isometric copies of all Banach spaces of density $\leqslant \aleph_1$.*
2. *V_{ω_1} is homogeneous with respect to its separable subspaces.*

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Projectional resolutions

Definition

A projectional resolution of the identity in a Banach space E is a transfinite sequence of norm one projections $\{P_\alpha\}_{\alpha<\kappa}$ such that

1. $\operatorname{dens} P_\alpha E < \operatorname{dens} E$.
2. $P_\alpha P_\beta = P_{\min\{\alpha,\beta\}}$.
3. $x = \lim_{\alpha\to\kappa} P_\alpha x$.
4. $P_\delta x = \lim_{\alpha\to\delta} P_\alpha x$ for every limit ordinal $\delta < \kappa$.

The notion of a PRI, originated by Lindenstrauss in the context of reflexive spaces, was studied by Fabian, Orihuela, Plichko, Valdivia, Zizler and others.

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Theorem

Assume $2^{\aleph_0} = \aleph_1$.

There exists a Banach space E *with a PRI* $\{P_\alpha\}_{\alpha<\omega_1}$ *and of density* $\aleph_1$, *which has the following properties:*

(a) *The family*

$$\{X \subseteq E \colon X \text{ is } 1\text{-complemented in } E\}$$

is, modulo linear isometries, the class of all Banach spaces of density $\leqslant \aleph_1$ *with a PRI.*

(b) *Given separable* 1*-complemented subspaces* $X, Y \subseteq E$, *given a linear isometry* $T \colon X \to Y$, *there exist a linear isometry* $H \colon E \to E$ *extending* T.

Moreover, these properties describe the space E *uniquely, up to a linear isometry.*

Valdivia compact spaces

Every compact space K embeds into a cube $[0,1]^\kappa$, for some κ.

A compact space K is Valdivia compact if it admits an embedding $e\colon K \to [0,1]^\kappa$ so that $e^{-1}[\Sigma(\kappa)]$ is dense in K, where $\Sigma(\kappa)$ is the Σ-product of κ copies of $[0,1]$, namely:

$$\Sigma(\kappa) = \{x \in [0,1]^\kappa \colon |\{\alpha \colon x(\alpha) \neq 0\}| \leqslant \aleph_0\}.$$

Theorem

Given a Valdivia compact space K of weight $\aleph_1$, the Banach space $C(K)$ has a PRI.

Theorem

Given a Banach space E with a PRI and of density $\aleph_1$, the space $\langle \overline{B}_{E^*}, \tau_{weakstar} \rangle$ is Valdivia compact.

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Theorem

Assume the Continuum Hypothesis. There exists a unique Valdivia compact space K of weight $\aleph_1$ with the following properties.

1. *Every Valdivia compact of weight $\leqslant \aleph_1$ is homeomorphic to a retract of K.*
2. *Given right-invertible maps $f\colon K \to X$ and $g\colon K \to Y$ onto compact metric spaces, given a homeomorphism $h\colon X \to Y$, there exists a homeomorphism $H\colon K \to K$ such that the square*

$$\begin{array}{ccc} K & \xrightarrow{H} & K \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Y \end{array}$$

is commutative.

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Theorem

There exists a linearly ordered Valdivia compact space K_{ω_1} such that for every nonempty linearly ordered Valdivia compact L there exists a continuous increasing surjection

$$K_{\omega_1} \longrightarrow\!\!\!\!\!\rightarrow L.$$

Main tool

The next notion, often used in domain theory, goes back to D. Scott, in the context of unsigned λ-calculus.

Embedding-Projection pairs

Fix a category $\mathfrak{K}$. We define a new category $\ddagger\mathfrak{K}$ as follows:

- The objects of $\ddagger\mathfrak{K}$ are the objects of $\mathfrak{K}$.
- An arrow from x to y in $\ddagger\mathfrak{K}$ is a pair $\langle e, p \rangle$, where $e \colon x \to y$ and $p \colon y \to x$ are $\mathfrak{K}$-arrows such that

$$p \circ e = \mathrm{id}_x.$$

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Example

Let $\mathfrak{K}$ be the category of separable Banach spaces with operators of norm $\leqslant 1$.

A Banach space of density $\aleph_1$ is the limit of a "continuous" ω_1-sequence in $\ddagger\mathfrak{K}$ if and only if it has a PRI.

Example

Let $\mathfrak{K}$ be the category of nonempty compact metric spaces with continuous maps.

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Further topics

- Complementably universal Banach space with a basis
- Approximate Fraïssé limits: The Gurarii space, the Urysohn space, etc.
- (with A. Avilés) Fraïssé functors and their limits, with applications to Banach spaces (in progress)

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