A SANDWICH THEOREM FOR CONVEXITY PRESERVING MAPS

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Abstract. We state a "sandwich" type theorem for maps of bi-convexity spaces (sets with two convexity structures). In a special case, this yields a result on separation of meet and join homomorphisms of distributive lattices.

1. Introduction

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *quasi-convex* (*quasi-concave*) if $f(x) \leqslant \max\{f(a), f(b)\}$ ($f(x) \geqslant \min\{f(a), f(b)\}$) whenever $x \in [a, b]$. A result of Förg-Rob, Nikodem and Páles [2] says that if $f, g\colon \mathbb{R} \to \mathbb{R}$ are such that f is quasi-concave, g is quasi-convex and $f \leqslant g$ then there exists a monotone map $h\colon \mathbb{R} \to \mathbb{R}$ such that $f \leqslant h \leqslant g$. A similar "sandwich" type theorem is valid for maps of arbitrary linearly ordered spaces (see below) but fails to hold for maps from $\mathbb{R}^2$ to $\mathbb{R}$.

One can easily check that a map $f\colon \mathbb{R} \to \mathbb{R}$ is quasi-convex (quasi-concave) iff $f^{-1}(-\infty, a]$ ($f^{-1}[a, +\infty)$) is convex for each $a \in \mathbb{R}$. Also, f is monotone iff it is simultaneously quasi-convex and quasi-concave. This suggests that the definition of such classes of maps can be formulated using the notion of convexity structures. Then one can look for sandwich type theorems for classes of spaces larger than linearly ordered sets. By a "space" we mean a *bi-convexity space*, i.e. a set equipped with two convexity structures - then an analogue of quasi-convex/quasi-concave maps can be defined. We show that a sandwich theorem for the class of S_4 bi-convexity spaces (which is the most natural class, containing real vector spaces) holds when the image space is a complete Boolean algebra (which is a bi-convexity space with the convexities consisting of ideals and filters). Under some natural conditions, complete Boolean algebras are the only spaces having such a property.

2. Bi-convexity spaces

By *a convexity* on a set X we mean, as in [8], a collection $\mathcal{G} \subset \mathcal{P}(X)$ stable under intersections, unions of chains and containing $\emptyset, X$. *The convex hull* of $A \subset X$ is defined by $\operatorname{conv}_{\mathcal{G}} A = \bigcap\{G \in \mathcal{G} : \ A \subset G\}$. *A bi-convexity space* is a triple $\langle X, \mathcal{L}, \mathcal{U}\rangle$, where $\mathcal{L}$ and $\mathcal{U}$ are two convexities on a set X, called *the lower* and *the upper convexity*. The members of $\mathcal{L}$ and $\mathcal{U}$ are called *lower convex* and *upper convex* respectively. We denote by conv_L and conv_U the convex hull with respect to the lower and the upper convexity respectively. The class of bi-convexity spaces was defined implicitly in [1]. Any subset of a bi-convexity space is also a bi-convexity space with the convexities consisting of the traces of lower/upper convex sets, see [8]. Every *convexity space* (a pair $\langle X, \mathcal{G}\rangle$, where $\mathcal{G}$ is a convexity on X) can be viewed as a bi-convexity space $\langle X, \mathcal{G}, \mathcal{G}\rangle$ - this provides natural examples of bi-convexity spaces. If $\langle X, \mathcal{L}, \mathcal{U}\rangle$ and $\langle X', \mathcal{L}', \mathcal{U}'\rangle$ are two bi-convexity spaces then a map $f\colon X \to X'$ is *lower*

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convexity preserving (*lcp* for short) provided $f^{-1}(G) \in \mathcal{L}$ whenever $G \in \mathcal{L}'$. Equivalently: $f(\operatorname{conv}_L S) \subset \operatorname{conv}_L f(S)$ for every finite $S \subset X$, see [8]. We define similarly the notion of an *upper convexity preserving* (*ucp*) map. A map $f\colon X \to X'$ is *convexity preserving* (*cp* for short) if f is both lcp and ucp. A bi-convexity space $\langle X, \mathcal{L}, \mathcal{U}\rangle$ is S_4 provided that for each two disjoint sets $L \in \mathcal{L}, U \in \mathcal{U}$ there exist an $H \in \mathcal{U}$ such that $U \subset H$, $L \subset X \setminus H$ and $X \setminus H \in \mathcal{L}$. Every S_4 space satisfies *the Pasch axiom*: for every $a \in X$ and for every $M_1, M_2, N_1, N_2 \in [X]^{<\omega}$ if $\operatorname{conv}_L(\{a\} \cup M_1) \cap \operatorname{conv}_U N_1 \neq \emptyset \neq \operatorname{conv}_L M_2 \cap \operatorname{conv}_U(\{a\} \cup N_2)$ then $\operatorname{conv}_L(N_1 \cup N_2) \cap \operatorname{conv}_U(M_1 \cup M_2) \neq \emptyset$ (indeed, otherwise there is no $H \in \mathcal{U}$ with $X \setminus H \in \mathcal{L}$ and $M_1 \cup M_2 \subset H$, $N_1 \cup N_2 \subset X \setminus H$). In fact, the Pasch axiom characterizes S_4, see [4]. Real vector spaces (or their convex subsets with the relative convexity) provide natural examples of S_4 convexity spaces.

Let $\langle L, \wedge, \vee\rangle$ be a lattice. Denote by $\mathcal{I}(L)$ ($\mathcal{F}(L)$) the collection of all ideals (filters) in L respectively ($\emptyset, L$ are non-proper filters and ideals). Let $\mathcal{G}(L)$ be the convexity generated by $\mathcal{I}(L) \cup \mathcal{F}(L)$. Then $\langle L, \mathcal{I}(L), \mathcal{F}(L)\rangle$ is a bi-convexity space and $\langle L, \mathcal{G}(L)\rangle$ is a convexity space (cf. [1, 8]). Note that the lower (upper) convex hull of a finite set $S \subset L$ equals the principal ideal (filter) generated by $\sup S$ ($\inf S$). By Stone-Birkhoff's theorem (cf. [3]), a lattice is S_4 iff it is distributive.

The sandwich theorem of Förg-Rob, Nikodem and Páles can be now stated as follows. For completeness, we sketch a proof.

Theorem 2.1. *Let X, Y be linearly ordered spaces. If $f, g\colon \langle X, \mathcal{G}(X)\rangle \to \langle Y, \mathcal{I}(Y), \mathcal{F}(Y)\rangle$ are such that f is upper convexity preserving, g is lower convexity preserving and $f \leqslant g$ then there exists a convexity preserving map $h\colon \langle X, \mathcal{G}(X)\rangle \to \langle Y, \mathcal{I}(Y), \mathcal{F}(Y)\rangle$ such that $f \leqslant h \leqslant g$.*

Proof. Set $I_1 = \{x \in X\colon\ (\forall\, y \leqslant x)\ f(y) \leqslant f(x)\}$ and $I_2 = \{x \in X\colon\ (\forall\, y \leqslant x)\ g(y) \geqslant g(x)\}$. Observe that $I_1, I_2 \in \mathcal{I}(X)$, $f \mid I_1$, $g \mid (X \setminus I_2)$ are increasing and $f \mid (X \setminus I_1)$, $g \mid I_2$ are decreasing. Now, if e.g. $I_1 \subset I_2$ then we can define $h(x) = g(x)$ for $x \in I_2$ and $h(x) = f(x)$ for $x \in X \setminus I_2$, obtaining a monotone map between f, g. □

Consider the four-element Boolean algebra $X = \{0,1\} \times \{0,1\}$ (with coordinate-wise order) and the three-element linearly ordered space $Y = \{0,1,2\}$. Define $f, g\colon X \to Y$ by setting $f(0,0) = f(0,1) = 0$, $f(1,0) = 1$, $f(1,1) = 2$, $g(0,0) = 0$, $g(1,0) = 1$ and $g(0,1) = g(1,1) = 2$. It is easy to check that f is ucp, g is lcp and $f \leqslant g$ but there is no cp map between f, g. Thus an analogue of Theorem 2.1 is not valid for non-linearly ordered spaces. This example can be easily modified to show the same for maps of type $\mathbb{R}^2 \to \mathbb{R}$.

3. Main result and its consequences

Let X be a bi-convexity space and let $\mathbb{B}$ be a Boolean algebra (always considered as a bi-convexity space). Furthermore, let $f, g\colon X \to \mathbb{B}$ be two maps such that f is ucp, g is lcp and $f \leqslant g$. We say that a map $h\colon M \to \mathbb{B}$, where $M \subset X$, is *well-placed* (*between f, g*) provided for each $S, T \in [M]^{<\omega}$ and $F, G \in [X]^{<\omega}$ the following implication holds:

$$\operatorname{conv}_L(S \cup G) \cap \operatorname{conv}_U(T \cup F) \neq \emptyset \implies \operatorname{conv}_L(h(S) \cup g(G)) \cap \operatorname{conv}_U(h(T) \cup f(F)) \neq \emptyset.$$

Observe that if $h\colon X \to \mathbb{B}$ is well-placed then $f \leqslant h \leqslant g$ and h is cp. Indeed, setting $S = F = \{p\}$ and $T = G = \emptyset$ above, we get $f(p) \leqslant h(p)$. Similarly $h(p) \leqslant g(p)$. If S is finite and $p \in \operatorname{conv}_L S$ then $\operatorname{conv}_L S \cap \operatorname{conv}_U\{p\} \neq \emptyset$ whence $\operatorname{conv}_L h(S) \cap \operatorname{conv}_U\{h(p)\} \neq \emptyset$. This means $h(p) \in \operatorname{conv}_L h(S)$. Thus h is lcp. By the dual argument, h is also ucp.

Lemma 3.1. *Let $\mathbb{B}$ be a complete Boolean algebra and let $\mathcal{A}, \mathcal{B}$ be two collections of finite subsets of $\mathbb{B}$ such that $\operatorname{conv}_L S \cap \operatorname{conv}_U T \neq \emptyset$ whenever $S \in \mathcal{A}, T \in \mathcal{B}$. Then $\bigcap_{S\in\mathcal{A}} \operatorname{conv}_L S \cap \bigcap_{T\in\mathcal{B}} \operatorname{conv}_U T \neq \emptyset$.*

Proof. Let $a_S = \sup S$, $b_T = \inf T$ and let $p = \sup_{T\in\mathcal{B}} b_T$. Then $b_T \leqslant p \leqslant a_S$ for every $S \in \mathcal{A}$, $T \in \mathcal{B}$. Hence $p \in \bigcap_{S\in\mathcal{A}} \operatorname{conv}_L S \cap \bigcap_{T\in\mathcal{B}} \operatorname{conv}_U T$. □

Lemma 3.2. *In every Boolean algebra, the following equivalence holds:*

$$\operatorname{conv}_L(A \cup B) \cap \operatorname{conv}_U(C \cup D) \neq \emptyset \iff \operatorname{conv}_L(A \cup \neg D) \cap \operatorname{conv}_U(C \cup \neg B) \neq \emptyset,$$

where $\neg X = \{\neg x :\ x \in X\}$ and $\neg x$ denotes the complement of x.

Proof. Suppose that $\operatorname{conv}_L(A \cup \neg D) \cap \operatorname{conv}_U(C \cup \neg B) = \emptyset$. Then there exists an ultrafilter P with $C \cup \neg B \subset P$ and $(A \cup \neg D) \cap P = \emptyset$. Now $D \subset P$ and B is disjoint from P. It follows that P separates $C \cup D$ from $A \cup B$ and consequently $\operatorname{conv}_L(A \cup B) \cap \operatorname{conv}_U(C \cup D) = \emptyset$. □

Now we can state our main result.

Theorem 3.3. *Let $\mathbb{B}$ be a complete Boolean algebra, let X be an S_4 bi-convexity space and let $f, g \colon X \to \mathbb{B}$ be such two maps that f is ucp, g is lcp and $f \leqslant g$. If $M \subset X$ then every well-placed map $h \colon M \to \mathbb{B}$ can be extended to a convexity preserving map $\overline{h} \colon X \to \mathbb{B}$ such that $f \leqslant \overline{h} \leqslant g$.*

Proof. We show that h can be extended to a well-placed map $\overline{h} \colon X \to \mathbb{B}$. The union of a chain of well-placed maps is also well-placed. Thus we should only show that for a fixed point $a \in X \setminus M$, h can be extended to a well-placed map $h' \colon M \cup \{a\} \to \mathbb{B}$. Consider two collections of intervals:

$$\begin{aligned}\mathcal{A}_U = \{&\operatorname{conv}_U(h(T) \cup f(F) \cup \neg h(S) \cup \neg g(G)) :\ S, T \in [M]^{<\omega}, F, G \in [X]^{<\omega},\\ &\qquad\qquad \operatorname{conv}_L(S \cup \{a\} \cup G) \cap \operatorname{conv}_U(T \cup F) \neq \emptyset\},\\ \mathcal{A}_L = \{&\operatorname{conv}_L(h(S) \cup g(G) \cup \neg h(T) \cup \neg f(F)) :\ S, T \in [M]^{<\omega}, F, G \in [X]^{<\omega},\\ &\qquad\qquad \operatorname{conv}_L(S \cup G) \cap \operatorname{conv}_U(T \cup \{a\} \cup F) \neq \emptyset\}.\end{aligned}$$

We show that every element of $\mathcal{A}_U$ meets every element of $\mathcal{A}_L$. Fix $S_1, S_2, T_1, T_2 \in [M]^{<\omega}$ and $F_1, F_2, G_1, G_2 \in [X]^{<\omega}$ such that

$$\operatorname{conv}_L(S_1 \cup \{a\} \cup G_1) \cap \operatorname{conv}_U(T_1 \cup F_1) \neq \emptyset \neq \operatorname{conv}_L(S_2 \cup G_2) \cap \operatorname{conv}_U(T_2 \cup \{a\} \cup F_2).$$

Applying the Pasch axiom for X we get $\operatorname{conv}_L(S_1 \cup G_1 \cup S_2 \cup G_2) \cap \operatorname{conv}_U(T_1 \cup F_1 \cup T_2 \cup F_2) \neq \emptyset$. Since h is well-placed, we obtain

$$\operatorname{conv}_L(h(S_1) \cup g(G_1) \cup h(S_2) \cup g(G_2)) \cap \operatorname{conv}_U(h(T_1) \cup f(F_1) \cup h(T_2) \cup f(F_2)) \neq \emptyset.$$

Applying Lemma 3.2 for $A = h(S_2) \cup g(G_2)$, $B = h(S_1) \cup g(G_1)$, $C = h(T_1) \cup f(F_1)$ and $D = h(T_2) \cup f(F_2)$ we get

$$\operatorname{conv}_L(h(S_2) \cup g(G_2) \cup \neg h(T_2) \cup \neg f(F_2)) \cap \operatorname{conv}_U(h(T_1) \cup f(F_1) \cup \neg h(S_1) \cup \neg g(G_1)) \neq \emptyset.$$

Now, applying Lemma 3.1 we can find a point

$$b \in \bigcap \mathcal{A}_U \cap \bigcap \mathcal{A}_L.$$

Define $h' \colon M \cup \{a\} \to \mathbb{B}$ by setting $h'(a) = b$ and $h' \mid M = h$. It remains to show that h' is well-placed. For let $S', T' \in [M \cup \{a\}]^{<\omega}$ and $F, G \in [X]^{<\omega}$ be such that $\operatorname{conv}_L(S' \cup G) \cap \operatorname{conv}_U(T' \cup F) \neq \emptyset$. We have to check that

$$(*) \qquad \operatorname{conv}_L(h'(S') \cup g(G)) \cap \operatorname{conv}_U(h'(T') \cup f(F)) \neq \emptyset.$$

If $S', T' \subset M$ or $a \in S' \cap T'$ then we are done, so suppose that e.g. $S' = S \cup \{a\}$ and $T' = T \subset M$. By the construction of $\mathcal{A}_U$, $b \in \operatorname{conv}_U(h(T) \cup f(F) \cup \neg h(S) \cup \neg g(G))$. Applying Lemma 3.2 for $A = \{b\}$, $B = h(S) \cup g(G)$, $C = h(T) \cup f(F)$ and $D = \emptyset$ we get $(*)$. This completes the proof. $\square$

Corollary 3.4. *Under the above assumptions, there exists a convexity preserving map $h \colon X \to \mathbb{B}$ with $f \leqslant h \leqslant g$.*

Proof. It is enough to show that the empty map is well-placed. Suppose that $p \in \operatorname{conv}_L G \cap \operatorname{conv}_U F \neq \emptyset$, where $F, G \in [X]^{<\omega}$. Since f is ucp and g is lcp, we have $f(p) \in \operatorname{conv}_U f(F)$ and $g(p) \in \operatorname{conv}_L g(G)$. Now, since $f(p) \leqslant g(p)$, we get $f(p), g(p) \in \operatorname{conv}_L g(G) \cap \operatorname{conv}_U f(F)$. $\square$

The next corollary was proved in [5], it is a Tietze type extension theorem for convexity preserving maps.

Corollary 3.5. *Let $\mathbb{B}$ be a complete Boolean algebra and let $h \colon A \to \mathbb{B}$ be a cp map defined on a convex subset of an S_4 convexity space $\langle X, \mathcal{G} \rangle$. Then there exists a cp map $\overline{h} \colon X \to \mathbb{B}$ extending h.*

Proof. Define $f, g \colon X \to \mathbb{B}$ by setting $f \mid A = g \mid A = h$ and $f \mid (X \setminus A) = 0_{\mathbb{B}}$, $g \mid (X \setminus A) = 1_{\mathbb{B}}$. We check that h is well-placed between f, g; then we can apply Theorem 3.3. Fix $S, T \in [A]^{<\omega}$ and $F, G \in [X]^{<\omega}$ with $p \in \operatorname{conv}(S \cup G) \cap \operatorname{conv}(T \cup F)$. If $F \cup G \subset A$ then $g(G) = h(G)$, $f(F) = h(F)$ and hence the set

$$K = \operatorname{conv}_L(h(S) \cup g(G)) \cap \operatorname{conv}_U(h(T) \cup f(F))$$

contains $h(p)$, since h is cp. Finally, if $G \setminus A \neq \emptyset$ then $1_{\mathbb{B}} \in g(G)$ and hence also $K \neq \emptyset$ since $\operatorname{conv}_L(h(S) \cup g(G)) = \mathbb{B}$. Similarly, $K \neq \emptyset$ in case $F \setminus A \neq \emptyset$. This completes the proof. $\square$

Remarks . (a) Observe that the statement of Corollary 3.4 for $\mathbb{B} = \{0, 1\}$ is in fact a reformulation of axiom S_4, since lower/upper convex sets can be identified with lcp/ucp characteristic functions.

(b) If f, g are as in Theorem 3.3 and h can be extended to a cp map $\overline{h}$ with $f \leqslant \overline{h} \leqslant g$ then h is well-placed between f, g. Indeed, if $x \in \operatorname{conv}_L(S \cup G) \cap \operatorname{conv}_U(T \cup F)$ in X then $\overline{h}(x) \in \operatorname{conv}_L(\overline{h}(S \cup G)) \cap \operatorname{conv}_U(\overline{h}(T \cup F)) \subset \operatorname{conv}_L(h(S) \cup g(G)) \cap \operatorname{conv}_U(h(T) \cup f(F))$.

(c) It is proved in [6] that complete Boolean algebras are the only S_4 convexity spaces satisfying the assertion of Corollary 3.5. Thus, if $\mathbb{B} = \langle B, \mathcal{L}, \mathcal{U} \rangle$ is a bounded partially ordered bi-convexity space satisfying the assertion of Theorem 3.3 and having some "reasonable" properties (specifically: the convexity generated by $\mathcal{L} \cup \mathcal{U}$ should be S_4 and the lower/upper convex hull of a set containing the greatest/least element should equal B), then we can deduce that $\mathbb{B}$ is isomorphic to a complete Boolean algebra, by showing that $\mathbb{B}$ satisfies the assertion of Corollary 3.5 and applying the mentioned result from [6].

Finally, we state a version of Theorem 3.3 for maps of distributive lattices.

Proposition 3.6. *Let K, L be lattices considered as bi-convexity spaces. A map $f\colon K \to L$ is lower convexity preserving if and only if it is a join homomorphism, i.e. $f(a \vee b) = f(a) \vee f(b)$ for every $a, b \in K$. Dually, f is ucp iff it is a meet homomorphism.*

Proof. Let f be lcp. Denote by I_p the principal ideal generated by a point $p \in L$. If $x, y \in K$ and $x \leqslant y$ then $y \in f^{-1}(I_{f(y)})$, hence $x \in f^{-1}(I_{f(y)})$ and consequently $f(x) \in I_{f(y)}$ which means $f(x) \leqslant f(y)$. Thus f is order-preserving. Now, for $x, y \in K$ we have $x, y \in f^{-1}(I_{f(x)\vee f(y)})$; hence also $x \vee y \in f^{-1}(I_{f(x)\vee f(y)})$ and consequently $f(x \vee y) \leqslant f(x) \vee f(y)$. Since f is order-preserving, we get $f(x) \vee f(y) = f(x \vee y)$.
Now let f be a join homomorphism and consider an ideal $I \subset L$. If $x, y \in f^{-1}(I)$ and $z \leqslant x \vee y$ then $f(z) \leqslant f(x \vee y) = f(x) \vee f(y) \in I$; hence $z \in f^{-1}(I)$. Thus $f^{-1}(I)$ is an ideal in K and therefore f is lcp. □

Theorem 3.7. *Let L be a distributive lattice, let $\mathbb{B}$ be a complete Boolean algebra. Furthermore, let $f, g\colon L \to \mathbb{B}$ be such two maps that f is a meet homomorphism, g is a join homomorphism and $f \leqslant g$. If K is a sublattice of L and $h\colon K \to \mathbb{B}$ is a lattice homomorphism satisfying*

$$\forall\, s, t \in K\ \forall\, a, b \in L,\ t \wedge a \leqslant s \vee b \implies h(t) \wedge f(a) \leqslant h(s) \vee g(b), \tag{1}$$

then there exists a lattice homomorphism $\overline{h}\colon L \to \mathbb{B}$ with $f \leqslant \overline{h} \leqslant g$ and $\overline{h} \mid K = h$. In particular, there exists a lattice homomorphism between f and g.

Proof. We should only check that h is well-placed. Let $S, T \in [K]^{<\omega}$, $F, G \in [L]^{<\omega}$ be such that $\operatorname{conv}_L(S \cup G) \cap \operatorname{conv}_U(T \cup F) \neq \emptyset$. This is equivalent to $(\inf T) \wedge (\inf F) \leqslant (\sup S) \vee (\sup G)$. Applying condition (1) we get $h(\inf T) \wedge f(\inf F) \leqslant h(\sup S) \vee g(\sup G)$ which means $\operatorname{conv}_L(h(S) \cup g(G)) \cap \operatorname{conv}_U(h(T) \cup f(F)) \neq \emptyset$. □

Observe that every partial lattice homomorphism satisfies condition (1) with $f = 0_{\mathbb{B}}$ and $g = 1_{\mathbb{B}}$. Thus, as a consequence, we obtain the theorem of Sikorski [7] on injectivity of complete Boolean algebras, which says that every partial Boolean homomorphism with values in a complete Boolean algebra can be extended to a full homomorphism. The theorem of Sikorski characterizes complete Boolean algebras among all distributive lattices. Thus there are no other bounded distributive lattices satisfying the assertion of Theorem 3.7 in place of $\mathbb{B}$.

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