Separation properties of convexity spaces

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Abstract

The purpose of this paper is to investigate some separation properties of sets with axiomatically defined convexity structures. We state a general separation theorem for pairs of convexities, improving some known results. As an application, we discuss separation properties of lattices, real vector spaces and modules.

Key words and phrases: convexity, geometrical space, Kakutani separation property, the Pasch axiom.

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1 Introduction

A classical theorem of Kakutani [8] says that two disjoint convex sets in a real vector space can be separated by a halfspace (i.e. a convex set with the convex complement). This theorem is also known as a geometric version of the Hahn-Banach theorem. In this paper we study the *Kakutani separation property* which means, roughly speaking, that disjoint sets of a certain type can be separated by disjoint complementary sets of the same type (the type will be described by means of convexity).

In 1952, J.W. Ellis [4] showed an abstract version of Kakutani's theorem, for pairs of convexities which are join-hull commutative (see the definitions below) and have a certain property, similar to the well-known Pasch axiom. The separation is realized by a set which is convex with respect to the first convexity and its complement is convex with respect to the second one. In case where both convexities are the same, Chepoi [2] showed in 1993 that the join-hull commutativity assumption is irrelevant.

Our purpose is to give some sufficient conditions for the Kakutani separation property of spaces with pairs of convexities defined by *finitary set operators* (see the definition below). We state a general separation theorem which is improves both the results of Ellis and Chepoi. In particular, we show that Ellis' theorem holds without the assumption of join-hull commutativity. We also characterize the Kakutani property of convexities in modules, defined by *algebraic intervals* [7]. This class of convexities contains the real vector space convexity, lattice convexities in Boolean algebras and other known convex structures.

2 Finitary set operators

Following van de Vel's monograph [15], by *a convexity* in a set X we mean a collection $\mathcal{G} \subset \mathcal{P}(X)$ containing $\emptyset, X$, closed under arbitrary intersections and closed under the unions of chains. *The convex hull* of a set $A \subset X$ is the set

$$\operatorname{conv} A = \bigcap\{G \in \mathcal{G} :\ A \subset G\}.$$

The convex hull of a set $\{x_1, \dots, x_n\}$ is called an *n-polytope* and is denoted by $[x_1, \dots, x_n]$. A 2-polytope $[a, b]$ is called *the segment joining* a, b. It is known [11] that $\operatorname{conv} A = \bigcup\{\operatorname{conv} F :\ F \text{ is a finite subset of } A\}$. A convexity $\mathcal{G}$ is called *N-ary* ($N \in \mathbb{N}$) if $A \in \mathcal{G}$ whenever $\operatorname{conv} F \subset A$ for all $F \in [A]^{\leqslant N}$, where $[A]^{\leqslant N}$ denotes the collection of at most N-element subsets of A. A space with 2-ary convexity will be called *geometrical*. A convexity $\mathcal{G}$ in X is *join-hull commutative* [9] provided $\bigcup_{g \in G}[g, x] \in \mathcal{G}$ whenever $G \in \mathcal{G}$ and $x \in X$. It is known [9, Th. 2] and easy to show that every join-hull commutative convexity is geometrical. For a general theory of convexity we refer to [15] and [11].

Let X be a set and let $D = [X]^{\leqslant N}$ or $D = [X]^{<\omega}$, where $[X]^{<\omega}$ denotes the collection of all finite subsets of X. Any map $r \colon D \to \mathcal{P}(X)$ such that $F \subset r(F)$ for all $F \in D$ will be called *a finitary set operator* (*FS-operator* for short) in X. An FS-operator $r \colon D \to \mathcal{P}(X)$ is *N-ary* provided $D \subset [X]^{\leqslant N}$. We say that a set $G \subset X$ is *r-convex* if for every $F \in D \cap \mathcal{P}(G)$ we have $r(F) \subset G$. Denote by $\mathcal{G}_r$ the collection of all r-convex subsets of X. Clearly, this is a convexity in X; $\mathcal{G}_r$ is N-ary if r is N-ary. We will say that a convexity $\mathcal{G}$ is *defined by* r if $\mathcal{G} = \mathcal{G}_r$. Every convexity is defined by its *polytope operator* $I = \operatorname{conv} \mid [X]^{<\omega}$. An N-ary convexity is defined by its *N-polytope operator* $I_N = \operatorname{conv} \mid [X]^{\leqslant N}$.

A set operator r is *transitive* provided for every $y \in r(\{x\} \cup F)$ we have $r(\{y\} \cup F) \subset r(\{x\} \cup F)$. A transitive 2-ary operator is called *an interval operator* (see [2]).

Let r be an FS-operator. We set $rA = \bigcup\{r(F) :\ F \in D \cap \mathcal{P}(A)\}$, $r^{n+1}A = r(r^n A)$, $r^0 A = A$.

Proposition 2.1. *If r is a finitary set operator in X then for every $A \subset X$ we have*

$$\operatorname{conv} A = \bigcup_{n\in\mathbb{N}} r^n A,$$

where conv *is the convex hull operator associated with convexity $\mathcal{G}_r$.*

Proof. Let $D = \operatorname{dom}(r)$, the domain of r. Denote by B the set on the right hand side. Clearly $A \subset B$ and $B \subset \operatorname{conv} A$. Let $F = \{a_1, \ldots, a_k\} \in D \cap \mathcal{P}(B)$, $a_i \in r^{n_i} A$. If we set $n = \max\{n_1, \ldots, n_k\}$ then $F \subset r^n A$ since $E \subset rE$ for $E \in D$ and thus $r(F) \subset r^{n+1} A \subset B$. Hence B is r-convex. $\square$

3 Main results

We start with a general separation theorem concerning two arbitrary convexities in a set.

Theorem 3.1. *Let $\mathcal{G}$ and $\mathcal{H}$ be two convexities in a set X. The following conditions are equivalent:*

(a) For every $x, y, z \in X$ and finite sets $S, T \subset X$ such that $x \in \operatorname{conv}_{\mathcal{G}}(\{z\} \cup S)$ and $y \in \operatorname{conv}_{\mathcal{H}}(\{z\} \cup T)$ it holds that $\operatorname{conv}_{\mathcal{G}}(\{y\} \cup S) \cap \operatorname{conv}_{\mathcal{H}}(\{x\} \cup T) \neq \emptyset$.

(b) If $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are disjoint then there exist disjoint sets $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $A \subset G$, $B \subset H$ and $G \cup H = X$.

Proof. (a) $\Longrightarrow$ (b) Let G be a maximal $\mathcal{G}$-convex set containing A disjoint from B and let H be a maximal $\mathcal{H}$-convex set containing B disjoint from G. We show that $G \cup H = X$. Suppose otherwise, i.e. there is some $z \in X \setminus (G \cup H)$. By the maximality of G we get $\operatorname{conv}_{\mathcal{G}}(G \cup \{z\}) \cap H \neq \emptyset$ and therefore there exists an $h \in H$ and a finite set $S \subset G$ with $h \in \operatorname{conv}_{\mathcal{G}}(\{z\} \cup S)$. By the same argument there is a $g \in G$ and a finite set $T \subset H$ with $g \in \operatorname{conv}_{\mathcal{H}}(\{z\} \cup T)$. By (a) we get $\operatorname{conv}_{\mathcal{G}}(\{g\} \cup S) \cap \operatorname{conv}_{\mathcal{H}}(\{h\} \cup T) \neq \emptyset$ which means that $G \cap H \neq \emptyset$; a contradiction.
(b) $\Longrightarrow$ (a) Let $A = \operatorname{conv}_{\mathcal{G}}(\{y\} \cup S)$ and $B = \operatorname{conv}_{\mathcal{H}}(\{x\} \cup T)$. Suppose that $A \cap B = \emptyset$. By (b) there are disjoint complementary sets $G \in \mathcal{G}$ and $H \in \mathcal{H}$ with $A \subset G$ and $B \subset H$. If $z \in G$ then $x \in \operatorname{conv}_{\mathcal{G}}(\{z\} \cup S) \subset G$. This yields that $z \notin G$. By the same argument $z \notin H$; a contradiction. $\square$

With respect to the above theorem we introduce the following definitions.

Definition 3.2. A pair (r, s) of FS-operators in X is said to satisfy *the Pasch axiom* provided for each $N, K < \omega$ with $[X]^{\leqslant N} \subset \operatorname{dom}(r)$, $[X]^{\leqslant K} \subset \operatorname{dom}(s)$ and for every $c, a_1, \dots, a_{N-1}, b_1, \dots, b_{K-1}, x, y \in X$ we have

$$\begin{aligned} & x \in r(c, a_1, \dots, a_{N-1}) \wedge y \in s(c, b_1, \dots, b_{K-1}) \implies \\ & \implies r(y, a_1, \dots, a_{N-1}) \cap s(x, b_1, \dots, b_{K-1}) \neq \emptyset. \end{aligned} \tag{P}$$

Here we write $r(x_1, \dots, x_N)$ and $s(x_1, \dots, x_K)$ instead of $r(\{x_1, \dots, x_N\})$ and $s(\{x_1, \dots, x_K\})$ respectively. If $r = s$ then we say that r satisfies *the Pasch axiom.*

Definition 3.3. A pair of convexities $(\mathcal{G}, \mathcal{H})$ in a set X has *the Kakutani separation property* provided for each two disjoint sets $A \in \mathcal{G}$ and $B \in \mathcal{H}$ there exist disjoint sets $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $A \subset G$, $B \subset H$ and $G \cup H = X$. If $\mathcal{G} = \mathcal{H}$ then we say that the convexity space $(X, \mathcal{G})$ has *the Kakutani separation property.*

Theorem 3.1 now says that the Kakutani separation property for two convexities is equivalent to the Pasch axiom for their polytope operators. A natural question is whether the Kakutani property holds for convexities defined by FS-operators satisfying the Pasch axiom. As we will show, the answer is affirmative for transitive set operators. We give also a similar result for arbitrary FS-operators but the Pasch axiom is then replaced with a more complicated formula.

Definition 3.4. We say that a pair (r, s) of FS-operators in X satisfies *axiom (Q)* provided for each natural numbers N, K with $[X]^{\leqslant N} \subset \operatorname{dom}(r)$, $[X]^{\leqslant K} \subset \operatorname{dom}(s)$ and for each $b, y_1, \dots, y_K, h_1, \dots, h_K, a_1, \dots, a_{N-1}$ we have

$$\begin{aligned} & b \in s(y_1, \dots, y_K) \wedge \forall i \leqslant K \; h_i \in r(y_i, a_1, \dots, a_{N-1}) \implies \\ & \implies r(b, a_1, \dots, a_{N-1}) \cap s(h_1, \dots, h_K) \neq \emptyset. \end{aligned} \tag{Q}$$

Setting $h_i = y_i$ for $i > 1$ above we see that (Q) implies the Pasch axiom.

Proposition 3.5. *If (r, s) is a pair of FS-operators satisfying the Pasch axiom and such that r is transitive then (r, s) satisfy (Q).*

Proof. Assume $b \in s(y_1, \dots, y_K)$ and $h_i \in r(y_i, a_1, \dots, a_{N-1})$ for $i \leqslant K$. By the Pasch axiom there exists $x_1 \in r(b, a_1, \dots, a_{N-1}) \cap s(h_1, y_2, \dots, y_K)$ and by the transitivity of r we have $r(x_1, a_1, \dots, a_{N-1}) \subset r(b, a_1, \dots, a_{N-1})$. Inductively, we can find $x_2, \dots, x_N$ such that $x_i \in s(h_1, \dots, h_i, y_{i+1} \dots, y_K)$ and $r(x_i, a_1, \dots, a_{N-1}) \subset r(b, a_1, \dots, a_{N-1})$. For $i = K$ we get $r(b, a_1, \dots, a_{N-1}) \cap s(h_1, \dots, h_K) \neq \emptyset$. ☐

We are going to show that axiom (Q) implies the Kakutani separation property. For convenience we will use the following abbreviations: $A_r[x] = \operatorname{conv}_r(A \cup \{x\})$ and $A_r^n[x] = r^n(A \cup \{x\})$ (the same for s). By Proposition 2.1 we have $A_r[x] = \bigcup_{n \in \mathbb{N}} A_r^n[x]$ (and the same for s).
In the next two lemmas we assume that (r, s) is a pair of FS-operators satisfying axiom (Q).

Lemma 3.6. *If H is s-convex and $r(x, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$, then for every $y \in H_s[x]$ we have $r(y, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$.*

Proof. The statement is clear for $y \in H_s^0[x] = H \cup \{x\}$, so assume that $r(y, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$ whenever $y \in H_s^n[x]$ and consider $y \in H_s^{n+1}[x]$. There are $y_1, \ldots, y_K$ in $H_s^n[x]$ with $y \in s(y_1, \ldots, y_K)$, where K is such that $[X]^{\leqslant K} \subset \mathrm{dom}(s)$. Now, by induction hypothesis, there exist $h_i \in r(y_i, a_1, \ldots, a_{N-1}) \cap H$, $i \leqslant K$. Applying (Q) we have $r(y, a_1, \ldots, a_{N-1}) \cap s(h_1, \ldots, h_K) \neq \emptyset$. Since H is s-convex, we get $r(y, a_1, \ldots, a_{N-1}) \cap H \neq \emptyset$. □

Lemma 3.7. *If G, H are such subsets of X that G is r-convex, H is s-convex and $G_r[x] \cap H \neq \emptyset \neq G \cap H_s[x]$ for some $x \in X$, then $G \cap H \neq \emptyset$.*

Proof. We use induction. Suppose that $G \cap H \neq \emptyset$ whenever G is r-convex, H is s-convex and $G_r^n[x] \cap H \neq \emptyset \neq G \cap H_s[x]$. Assume $G_r^{n+1}[x] \cap H \neq \emptyset \neq G \cap H_s[x]$. This means that $r(u_1, \ldots, u_N) \cap H \neq \emptyset$ for some $u_1, \ldots, u_N \in G_r^n[x]$, where N is such that $[X]^{\leqslant N} \subset \mathrm{dom}(r)$. Now, for $i \leqslant N$, we have

$$G_r^n[x] \cap H_s[u_i] \neq \emptyset \neq G \cap (H_s[u_i])_s[x].$$

By induction hypothesis there are $g_i \in G \cap H_s[u_i]$ for $i \leqslant N$. Hence, as $r(u_1, \ldots, u_N) \cap H \neq \emptyset$ and $g_1 \in H_s[u_1]$, applying Lemma 3.6 we see that $H \cap r(g_1, u_2, \ldots, u_N) \neq \emptyset$. Inductively, using Lemma 3.6, we infer that $H \cap r(g_1, \ldots, g_i, u_{i+1}, \ldots, u_N) \neq \emptyset$ for $i = 1, \ldots, N$. Particularly we get $H \cap r(g_1, \ldots, g_N) \neq \emptyset$ which means that G and H intersect, since G is r-convex. □

Now we can state the main result.

Theorem 3.8. *If (r, s) is a pair of finitary set operators satisfying axiom (Q) then the pair of convexities $(\mathcal{G}_r, \mathcal{G}_s)$ has the Kakutani separation property.*

Proof. Let G be a maximal r-convex set containing A disjoint from B and let H be a maximal s-convex set containing B disjoint from G. Observe that $G \cup H = X$. Indeed, otherwise there exists $x \in X \setminus (G \cup H)$ and by the maximality of G and H we obtain $G_r[x] \cap H \neq \emptyset \neq G \cap H_s[x]$. Hence, in view of Lemma 3.7, the sets G, H intersect; a contradiction. □

4 Consequences

Theorem 4.1. *Let $\mathcal{G}, \mathcal{H}$ be two 2-ary convexities in a set X. The following conditions are equivalent:*

(a) If $a_1 \in [c, a]_{\mathcal{G}}$ and $b_1 \in [c, b]_{\mathcal{H}}$ then $[a, b_1]_{\mathcal{G}} \cap [b, a_1]_{\mathcal{H}} \neq \emptyset$.

(b) If $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are disjoint then there exist disjoint sets $G \in \mathcal{G}$, $H \in \mathcal{H}$ such that $A \subset G$, $B \subset H$ and $G \cup H = X$.

Proof. Implication (a) $\Longrightarrow$ (b) follows from Theorem 3.8 and Proposition 3.5 by setting $r(a,b) = [a,b]_{\mathcal{G}}$ and $s(a,b) = [a,b]_{\mathcal{H}}$ since polytope maps are transitive. Implication (b) $\Longrightarrow$ (a) follows from Theorem 3.1. □

J.W. Ellis has proved in [4] that the Kakutani separation property holds for two join-hull commutative convexities such that their segments satisfy condition (a) above. As every join-hull commutative convexity is 2-ary, Theorem 4.1 improves the result of Ellis.

We say that two sets $A, B \subset X$ are *screened* by C, D if $A \subset C \setminus D$, $B \subset D \setminus C$ and $C \cup D = X$ (cf. [14]).

Theorem 4.2. *Let X be a space with N-ary convexity. The following conditions are equivalent:*

(i) Every two disjoint convex subsets of X can be separated by complementary halfspaces.

(ii) Every two disjoint N-polytopes can be separated by complementary halfspaces.

(iii) Every two disjoint N-polytopes can be screened by convex sets.

(iv) If $x \in [c, a_1, \dots, a_{N-1}]$, $y \in [c, b_1, \dots, b_{N-1}]$ then $[y, a_1, \dots, a_{N-1}] \cap [x, b_1, \dots, b_{N-1}] \neq \emptyset$.

Proof. Implication (iv) $\Longrightarrow$ (i) follows from Theorem 3.8 by setting $r = s = I_N$ (the N-polytope map). Since implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial, it remains to show that (iii) $\Longrightarrow$ (iv).

We use the same argument as in the proof of Theorem 3.1. Suppose (iv) fails for some x, y, c, $a_1, \dots, a_{N-1}$ and $b_1, \dots, b_{N-1}$. Let C, D be two convex sets screening $[y, a_1, \dots, a_{N-1}]$ and $[x, b_1, \dots, b_{N-1}]$. If $c \in C$ then $x \in [c, a_1, \dots, a_{N-1}] \subset C$, since C is convex. Hence $c \notin C$; by the same argument $c \notin D$. It follows that $C \cup D \neq X$; a contradiction. □

V. Chepoi has stated in [2] the equivalence of (i), (ii) and the property of N-polytopes which is in fact a reformulation of axiom (Q). Thus Theorem 4.2 improves the result of Chepoi.

Theorem 4.3 (Chepoi [2])**.** *Let X be a geometrical space defined by an interval operator $I \colon [X]^{\leqslant 2} \to \mathcal{P}(X)$. If I satisfies the Pasch axiom then X has the Kakutani separation property.*

Proof. This follows immediately from Theorem 3.8 and Proposition 3.5 by setting $r = s = I$. □

5 Examples and applications

The convexity in a lattice. Let L be a lattice; we denote by ab and $a+b$ the infimum and the supremum of a,b in L, respectively. Set $[a,b]=\{x\in L:\ ab\leqslant x\leqslant a+b\}$. This defines a convexity in L (see [1], [13], [14]) and L is a geometrical space with such a convexity. Observe ideals and filters are convex, more generally, a convex set is an order-convex sublattice. It is easy to check that a proper halfspace is either a prime filter or a prime ideal (see [13]). Recall that a lattice L is *distributive* provided $a(b+c)=(ab)+(ac)$ for every $a,b,c\in L$.

Using Theorem 4.3 we are able to give a short proof of Stone-Birkhoff's separation theorem.

Theorem 5.1. *A lattice has the Kakutani property iff it is distributive.*

Proof. Let L be a lattice, assume first that L has the Kakutani property. Suppose that there are $a,b,c\in L$ with $x=(ab)+(ac)<a(b+c)=y$. Thus $y\notin[ab,ac]$ so there exists a halfspace $H\subset L$ with $y\notin H\supset[ab,ac]$. Observe that $a\in L\setminus H$ since $x\in H$ and $y\in[x,a]\cap(L\setminus H)$. Hence $b,c\in H$ since $ab\in H\cap[a,b]$ and $ac\in H\cap[a,c]$. Now $b+c\in H$ and $y\in[x,b+c]\subset H$ which gives a contradiction.

Now assume that L is distributive and consider $a,a_1,b,b_1,c\in L$ with $a_1\in[c,a]$ and $b_1\in[c,b]$. We have $ac\leqslant a_1\leqslant a+c$ and $bc\leqslant b_1\leqslant b+c$, whence

$$a_1b\leqslant(a+c)b=(ab)+(cb)\leqslant a+b_1. \tag{1}$$

Similarly $ab_1\leqslant a_1+b$. We set $x=(a+b_1)(a_1+b)$. Clearly $x\leqslant a_1+b$ and by (1) we get $x\geqslant a_1b(a_1+b)=a_1b$. Hence $x\in[a_1,b]$. By the same argument $x\in[a,b_1]$. Now Theorem 4.3 implies that L has the Kakutani property. □

Geometrical modules. Let us consider a ring R (with unity, but not necessarily commutative). Following Jamison [7] we say that a subset J of R is *an algebraic interval* provided

(i) $0,1\in J$ and

(ii) $\alpha,\beta,\gamma\in J$ implies $\gamma\alpha+(1-\gamma)\beta\in J$.

Fix an R-module M. For $a,b\in M$ we set

$$[a,b]_J=\{\lambda a+(1-\lambda)b:\ \lambda\in J\}.$$

This defines a 2-ary convexity $\mathcal{G}_J$ in M; elements of $\mathcal{G}_J$ will be called J-*convex.* Observe that (i) and (ii) imply that the set $[a,b]_J$ defined above is in fact the segment joining a,b with respect to $\mathcal{G}_J$. Any module over a ring R with such defined convexity will be called *a geometrical module over* (R,J). In particular R

alone is a geometrical module over itself. Observe that J is the segment joining $0, 1 \in R$.

We shall state an algebraic condition for geometrical modules equivalent to the Kakutani property. First we present some examples of geometrical modules.

Examples 5.2. (a) Let $\mathbb{R}$ denote the field of reals. If $J = [0, 1]$ then the convexity $\mathcal{G}_J$ in a real vector space is the usual one. If $J = [0, 1] \cap \mathbb{Q}$ then it is the *rational* convexity. Finally, $J = \{\frac{k}{2^n} \in [0, 1] : \ k, n \in \mathbb{N}\}$ defines the *Jensen* convexity (see also [3], [5]).

(b) Let B be a Boolean algebra and let $J = B$. Denote by $\triangle$ the symmetric difference in B. Then $(B, \triangle, *, 0, 1)$ is a (Boolean) ring. One can easily check that $[a, b]_J = \{x \in B : \ ab \leqslant x \leqslant a + b\}$. Hence this convexity is the same as the convexity defined by the lattice structure in B.

(c) Let R be the ring of all real measurable functions defined on a measurable space $(T, \mathfrak{M})$. We set $J = \{\chi_A : \ A \in \mathfrak{M}\}$ where χ_A denotes the characteristic function of A. It is easily seen that $\chi_C \chi_A + (1 - \chi_C)\chi_B = \chi_G$ where $G = (A \cap C) \cup (B \setminus C)$, hence J is an algebraic interval. A J-convex set is called *decomposable*; decomposable subsets of $L^1(\mu)$, the space of μ-integrable real functions, were introduced in [6].

Theorem 5.3. *Let R be a ring and $J \subset R$. The following conditions are equivalent:*

(a) Every geometrical module over (R, J) has the Kakutani property.

(b) J (with the relative convexity) has the Kakutani property.

(c) For each $\alpha, \beta \in J$ there exists $\gamma \in J$ with $(1 - \alpha)\beta = \gamma(1 - \alpha\beta)$.

Proof. (a) $\Longrightarrow$ (b) As we have already mentioned, J is a convex subset of R. Hence disjoint convex subsets of J are convex in R and the Kakutani property of J follows from that of R.

(b) $\Longrightarrow$ (c) Fix $\alpha, \beta \in J$. Notice that $\alpha\beta \in [0, \beta]_J \subset J$. Suppose that $\beta \notin [1, \alpha\beta]_J$. As $\{\beta\}$ and $[1, \alpha\beta]_J$ are convex and disjoint, there exists a halfspace $H \subset J$ with $\beta \notin H$ and $[1, \alpha\beta]_J \subset H$. If $0 \in H$ then $\beta = \beta \cdot 1 + (1 - \beta) \cdot 0 \in H$, if $0 \notin H$ then $\alpha\beta = \alpha\beta + (1 - \alpha) \cdot 0 \notin H$. Both cases yield a contradiction. It follows that $\beta \in [1, \alpha\beta]_J$, i.e. there exists $\gamma \in J$ with $\beta = \gamma + (1 - \gamma)\alpha\beta$. Modifying this expression we get $(1 - \alpha)\beta = \gamma(1 - \alpha\beta)$.

(c) $\Longrightarrow$ (a) In view of Theorem 4.3 we must verify the Pasch axiom. Fix a geometrical module M over (R, J) and $a, a_1, b, b_1, c \in M$ such that $a_1 \in [c, a]_J$ and $b_1 \in [c, b]_J$. We have

$$a_1 = \alpha a + (1 - \alpha)c, \qquad b_1 = \beta b + (1 - \beta)c,$$

for some $\alpha, \beta \in J$. Let $\gamma \in J$ be as in condition (c), i.e.

$$\beta = \alpha\beta + \gamma(1 - \alpha\beta) = (1 - \gamma)\alpha\beta + \gamma. \tag{1}$$

We set

$$\delta = (1-\gamma)\alpha. \tag{2}$$

Observe that $\delta \in [0,\alpha]_J \subset J$ and $\gamma = (1-\delta)\beta$. Now take

$$x_1 = (1-\gamma)a_1 + \gamma b, \qquad x_2 = (1-\delta)b_1 + \delta a.$$

Notice that $x_1 \in [a_1, b]_J$ and $x_2 \in [a, b_1]_J$. It remains to check that $x_1 = x_2$. Let us compute the difference $x_1 - x_2$. We have

$$\begin{aligned} x_1 - x_2 &= (1-\gamma)\Big(\alpha a + (1-\alpha)c\Big) + \gamma b - (1-\delta)\Big(\beta b + (1-\beta)c\Big) - \delta a \\ &= \delta a + (1-\gamma)(1-\alpha)c + \gamma b - \gamma b - (1-\delta)(1-\beta)c - \delta a \\ &= \Big((1-\gamma)(1-\alpha) - (1-\delta)(1-\beta)\Big)c. \end{aligned}$$

Now, applying (1) and (2) we see that

$$\begin{aligned} (1-\delta)(1-\beta) &= 1 - \delta - \beta + (1-\gamma)\alpha\beta = 1 - \delta - \beta + \beta - \gamma \\ &= 1 - \gamma - (1-\gamma)\alpha = (1-\gamma)(1-\alpha). \end{aligned}$$

Hence $x_1 = x_2$ and this completes the proof. □

Corollary 5.4. (a) *If $J = F \cap [0,1]$ where F is a subfield of reals then every real vector space with J-convexity has the Kakutani property.*
(b) *(Palés [10]) The Jensen convexity in $\mathbb{R}$ does not have the Kakutani property.*
(c) *For every measure μ, the ring of μ-measurable real functions, with the convexity of decomposable sets, has the Kakutani property. The same is true for $L^1(\mu)$.*

Proof. (a) is clear, by condition (c) of Theorem 5.3.
(b) follows from the equality $(1-\frac{1}{2})\frac{1}{2}\Big/(1-\frac{1}{2}\frac{1}{2}) = \frac{1}{3}$.
(c) Equality $(1-\chi_A)\chi_B = \gamma(1-\chi_A\chi_B)$ is valid for $\gamma = \chi_B$. Moreover, $L^1(\mu)$ is a decomposable subset of the ring of all μ-measurable functions. □

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