EXTENSION THEOREMS IN AXIOMATIC THEORY OF CONVEXITY

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Abstract. We present a criterion for extending convexity preserving maps of convexity spaces. In a special case of convexity generated by a lattice structure this gives the Sikorski's Extension Criterion for extending of maps of lattices. We also consider the class of convexity absolute extensors. It appears that complete Boolean algebras with a natural convexity belong to this class. In particular, we present an analogue of Tietze-Urysohn's Extension Theorem for maps of convexity spaces with values in a complete Boolean algebra.

1. General definitions

By *a convexity space* we mean a set X together with a collection $\mathcal{G} \subset \mathcal{P}(X)$ satisfying the following axioms:

(1) $\emptyset, X \in \mathcal{G}$,
(2) $\bigcap \mathcal{A} \in \mathcal{G}$ for nonempty $\mathcal{A} \subset \mathcal{G}$,
(3) $\bigcup \mathcal{A} \in \mathcal{G}$ whenever $\mathcal{A} \subset \mathcal{G}$ is a chain.

Elements of $\mathcal{G}$ we call *convex sets*. For $A \subset X$ we define *the convex hull of* A as follows

$$\operatorname{conv} A = \bigcap \{G \in \mathcal{G} : \ A \subset G\}.$$

We will write $[a, b]$ instead of $\operatorname{conv}\{a, b\}$. One can prove that $\operatorname{conv} A = \bigcup \{\operatorname{conv} F : \ F \in [A]^{<\omega}\}$ (see [5]). The most important class of convexity spaces seems to be the class of so called *spaces of arity two*, i.e. spaces satisfying

(3') If $A \subset X$ and for every $a, b \in A$ there exists a $G \in \mathcal{G}$ with $a, b \in G \subset A$ then $A \in G$,

instead of (3), which is stronger than (3). Clearly, condition (3') can be formulated also in the form: $A \in \mathcal{G}$ whenever for every $a, b \in A$, $[a, b] \subset A$. Throughout this paper a convexity space of arity two will be called briefly *a geometrical space*. The convexity of a geometrical space is said to be *an interval convexity*, see [1]. For a systematic study of the theory of convexity see e.g. van de Vel [8] or Soltan [5].

A map $f : X \to Y$ of convexity spaces is called *convexity preserving* (*cp-map* for short) provided $f^{-1}(G)$ is convex in X whenever G is convex in Y. This is equivalent to the condition $f(\operatorname{conv} A) \subset \operatorname{conv} f(A)$ for any finite $A \subset X$ and, in the class of geometrical spaces, to $f([a, b]) \subset [f(a), f(b)]$ for $a, b \in X$ (see [8]).

A convexity space is called S_3 provided one-element subsets are convex and for every $x \in X$ and a finite subset $F \subset X$ with $x \notin \operatorname{conv} F$ there exists a halfspace H (i.e. a convex set with convex complement) such that $x \notin H$ and $F \subset H$. This property implies that a point can be separated

Date: November, 1997.
1991 *Mathematics Subject Classification.* Primary: 52A01, Secondary: 06B05.
Key words and phrases. convexity, geometrical space, cp-map, lattice.

from *any* convex set by a halfspace (see [5]). Finally, a convexity space is S_4 if points are convex and any two disjoint convex sets can be separated by a halfspace.

The convexity in a lattice. Let L be a lattice. For $a, b \in L$ we set $I(a, b) = \{x \in L : ab \leqslant x \leqslant a + b\}$, where ab and $a + b$ denote the infimum and the supremum of a, b respectively. A subset $G \subset L$ is *convex* provided $I(a, b) \subset G$ whenever $a, b \in A$. This defines a convexity in L (called *the convexity of a lattice*), the set L endowed with this convexity is a space of arity two (see [6], [7]) and $[a, b] = I(a, b)$ for $a, b \in L$. A lattice is distributive iff it is S_3 (see [7] or [2]). A subset G of a lattice L is convex iff $G = I \cap F$ where I is an ideal and F is a filter in L (see [6]).

We will consider only bounded lattices, with the least element 0 and the greatest element 1. If $S = \{a_1, \ldots, a_n\}$ is a subset of a bounded lattice then

$$\operatorname{conv} S = [a_1 \ldots a_n, a_1 + \cdots + a_n] = \operatorname{conv}(S \cup \{0\}) \cap \operatorname{conv}(S \cup \{1\}).$$

It is well-known that a map f of bounded lattices is a homomorphism if and only if f is a cp-map, $f(0) = 0$ and $f(1) = 1$ (see [7, 3.5.4]).

An important example of a (distributive) lattice is a Boolean algebra and, in particular, a power set $\mathcal{P}(S)$ for any set S.

2. Extension Criterion

Condition (I). Let X, Y be two convexity spaces and let $f : M \to Y$ be a map, $M \subset X$. We say that f *has property (I)* provided for every $n \in \mathbb{N}$, $S_1, \ldots, S_n \in [M]^{<\omega}$

$$\operatorname{conv} S_1 \cap \cdots \cap \operatorname{conv} S_n \neq \emptyset \implies \operatorname{conv} f(S_1) \cap \cdots \cap \operatorname{conv} f(S_n) \neq \emptyset$$

(here "conv" means the convex hull in X and Y respectively).

Clearly, if f can be extended to a cp-map over X then f has to satisfy the condition (I). Our purpose is to show that, in some situations, the converse is also true.

Let X be a convexity space, $a, b, c \in X$. Any member of $[a, b] \cap [a, c] \cap [b, c]$ is called *a median* of a, b, c. A subset $A \subset X$ is *median-stable* if for every $a, b, c \in A$ every median of a, b, c belongs to A (see [7]). We say that $M \subset X$ *generates* X if X is the only median-stable subset containing M.

Lemma 2.1. *Let Y be an S_3 convexity space and let y be a median of $a, b, c \in Y$. Then for every $S \subset Y$ the following holds true:*

$$\operatorname{conv}(S \cup \{y\}) = \operatorname{conv}(S \cup \{a, b\}) \cap \operatorname{conv}(S \cup \{a, c\}) \cap \operatorname{conv}(S \cup \{b, c\}).$$

Proof. Denote by A the set on the right hand side of the above equality. Clearly $S \cup \{y\} \subset A$ and hence $\operatorname{conv}(S \cup \{y\}) \subset A$. Assume $z \notin \operatorname{conv}(S \cup \{y\})$. As Y is S_3, there exists a halfspace H with $z \notin H \supset \operatorname{conv}(S \cup \{y\})$. Observe that $|H \cap \{a, b, c\}| \geqslant 2$ since $y \in H$ and $X \setminus H$ is convex. Now, if for instance $a, b \in H$ then we get $\operatorname{conv}(S \cup \{a, b\}) \subset H$ and therefore $z \notin A$. □

Now we are ready to prove the extension theorem, which is an analogue of Sikorski's Extension Criterion.

Theorem 2.2. *Let X be a convexity space and let M generate X. If $f : M \to Y$ is a map satisfying condition (I) and Y is S_3, then there exists a unique convexity preserving map $g : X \to Y$ extending f.*

Proof. Clearly, the union of a chain of maps with the property (I) satisfies (I) as well. Thus, by the Lemma of Kuratowski-Zorn, there exists a maximal map g extending f and satisfying (I). Suppose that $\mathrm{dom}(g) \neq X$, where $\mathrm{dom}(g)$ denotes the domain of g. As M generates X and $M \subset \mathrm{dom}(g)$, $\mathrm{dom}(g)$ is not median-stable, which means that there are $a, b, c \in \mathrm{dom}(g)$ and its median $x \notin \mathrm{dom}(g)$. Since g satisfies condition (I), there exists a point $y \in [g(a), g(b)] \cap [g(a), g(c)] \cap [g(b), g(c)]$. We set $N = \mathrm{dom}(g) \cup \{x\}$ and define $g_1 \colon N \to Y$ by $g_1 \mid \mathrm{dom}(g) = g$ and $g_1(x) = y$. To get a contradiction it is enough to show that g_1 satisfies (I).

Assume $S_1, \dots, S_n \in [N]^{<\omega}$ and $\mathrm{conv}\, S_1 \cap \dots \cap \mathrm{conv}\, S_n \neq \emptyset$. If $x \in S_i$ then setting $T_i = S_i \setminus \{x\}$, by the choice of x we get

$$\mathrm{conv}\, S_i \subset \mathrm{conv}(T_i \cup \{a, b\}) \cap \mathrm{conv}(T_i \cup \{a, c\}) \cap \mathrm{conv}(T_i \cup \{b, c\}).$$

On the other hand, as $g_1(x)$ is a median of $g_1(a), g_1(b), g_1(c)$, by Lemma 2.1 we have

$$\mathrm{conv}\, g_1(S_i) = \mathrm{conv}\, g(T_i \cup \{a, b\}) \cap \mathrm{conv}\, g(T_i \cup \{a, c\}) \cap \mathrm{conv}\, g(T_i \cup \{b, c\}).$$

Hence, we can replace every S_i containing x by the sets $T_i \cup \{a, b\}, T_i \cup \{a, c\}, T_i \cup \{b, c\}$ which do not contain x, preserving the intersection $\mathrm{conv}\, g_1(S_1) \cap \dots \cap \mathrm{conv}\, g_1(S_n)$. Now the property (I) for g_1 follows from that of g. Thus we have proved that $\mathrm{dom}(g) = X$.

Now observe that g is cp. Indeed, if $s \in \mathrm{conv}\, S$ and $S \in [X]^{<\omega}$ then $\mathrm{conv}\{s\} \cap \mathrm{conv}\, S \neq \emptyset$ and, by condition (I), $\mathrm{conv}\{g(s)\} \cap \mathrm{conv}\, g(S) \neq \emptyset$ which means $g(s) \in \mathrm{conv}\, g(S)$.

The uniqueness of g follows from the fact that for two cp-maps $g_1, g_2 \colon X \to Y$ the set $\{x \in X : g_1(x) = g_2(x)\}$ is median-stable. This completes the proof. □

3. An Application to the lattice theory

The convexity of a lattice is *binary* in the following sense: if $\mathcal{A}$ is a finite family of convex sets such that each two of its members intersect, then $\bigcap \mathcal{A} \neq \emptyset$ (see [7]). Thus, for a mapping f with values in a lattice, the condition (I) is equivalent to the following

(I') For every $S_1, S_2 \in [\mathrm{dom}(f)]^{<\omega}$ if $\mathrm{conv}\, S_1 \cap \mathrm{conv}\, S_2 \neq \emptyset$ then $\mathrm{conv}\, f(S_1) \cap \mathrm{conv}\, f(S_2) \neq \emptyset$.

If M is a subset of a lattice L and $0, 1 \in M$ then M generates L in the sense of our definition if M generates L as a lattice. Indeed, ab is a median of $0, a, b$ and $a + b$ is a median of $a, b, 1$.

As a consequence of Theorem 2.2 we get the following generalization of the Sikorski Extension Criterion.

Theorem 3.1. *Let K and L be bounded lattices, let L be distributive and let M generate K as a lattice. A map $f \colon M \to L$ can be (uniquely) extended to a homomorphism if and only if for every $a_1, \dots, a_n, b_1, \dots, b_m \in M$ the following condition holds true:*

$$(*) \qquad a_1 \dots a_n \leqslant b_1 + \dots + b_m \implies f(a_1) \dots f(a_n) \leqslant f(b_1) + \dots + f(b_m).$$

Proof. Suppose f satisfies condition (*). We can assume that $0, 1 \in M$ and $f(0) = 0$, $f(1) = 1$. Now, (*) means that $\mathrm{conv}\{0, b_1, \dots, b_m\} \cap \mathrm{conv}\{a_1, \dots, a_n, 1\} \neq \emptyset$ implies

$$\mathrm{conv}\, f(\{0, b_1, \dots, b_m\}) \cap \mathrm{conv}\, f(\{a_1, \dots, a_n, 1\}) \neq \emptyset.$$

In view of our remarks above, this is equivalent to the property (I). Hence f can be uniquely extended to a cp-map $g \colon K \to L$ (since M generates K). Finally g is a homomorphism as it is cp and preserves $0, 1$. Obviously, condition (*) is necessary. □

Corollary 3.2 (Sikorski Extension Criterion [4])**.** *Let A, B be two Boolean algebras and let M generate A as a Boolean algebra. A map $f\colon M \to B$ can be uniquely extended to a homomorphism iff for every $a_1, \dots, a_n, b_1, \dots, b_m \in M$ an equality*

$$a_1 \dots a_n \neg b_1 \dots \neg b_m = 0$$

implies

$$f(a_1) \dots f(a_n) \neg f(b_1) \dots \neg f(b_m) = 0.$$

Proof. Define $M' = M \cup \{\neg x : \ x \in M\}$ and extend f to a map $f'\colon M' \to B$ by setting $f'(\neg x) = \neg f(x)$ for $x \in M$ (if both $x, \neg x \in M$ then $f(x)f(\neg x) = 0$ and $\neg f(x) \neg f(\neg x) = 0$ which means $f(\neg x) = \neg f(x)$). Now, M' generates A as a lattice and f' satisfies condition (*) of Theorem 3.1. Using Theorem 3.1 we obtain the desired extension of f. $\square$

4. Extension theorem for S_4 convexity spaces

Now we investigate the problem of extending maps with the condition (I) defined on subsets of S_4 convexity spaces. If we want to extend such maps over the whole space, the image space must have some additional properties. Let us introduce the notation analogue to that of the (topological) extension theory: we will say that a convexity space Y is *a convexity absolute extensor* (*CAE* for short) provided for every subset M of an S_4 convexity space X every map $f\colon M \to Y$ with the property (I) can be extended to a convexity preserving map $g\colon X \to Y$.

As we have mentioned above, every Boolean algebra is a convexity space with the convexity of a lattice. If α is an element of a set S then we denote by α^+ the principal ultrafilter in $\mathcal{P}(S)$ generated by α, namely $\alpha^+ = \{A \subset S : \ \alpha \in A\}$. Our aim is to show that every complete Boolean algebra is a CAE.

We shall start with a couple of lemmas.

Lemma 4.1. *The two-element lattice $\{0, 1\}$ is a convexity absolute extensor.*

Proof. Let $f\colon M \to \{0, 1\}$ satisfy (I), we set $A_i = f^{-1}(i)$, $i = 0, 1$. The condition (I) means that $\operatorname{conv} A_0 \cap \operatorname{conv} A_1 = \emptyset$. Hence, as X is S_4, there exists a halfspace H with $A_1 \subset H$ and $A_0 \cap H = \emptyset$. One can easily check that the characteristic function of H is the desired extension of f. $\square$

Lemma 4.2. *Let S be a set. The collection*

$$\mathcal{B} = \{\alpha^+ : \ \alpha \in S\} \cup \{\mathcal{P}(S) \setminus \alpha^+ : \ \alpha \in S\}$$

is a subbase of the convexity space $\mathcal{P}(S)$, i.e. the convexity of $\mathcal{P}(S)$ is the least one which contains $\mathcal{B}$.

Proof. In view of [8, Proposition I.1.7.3] it is enough to show that for any $x \notin \operatorname{conv}\{a_1, \dots, a_n\} = P$ there exists an $\alpha \in S$ such that α^+ and $\alpha^- = \mathcal{P}(S) \setminus \alpha^+$ separate x from P. We have $P = [c, d]$ where $c = a_1 \dots a_n$ and $d = a_1 + \dots + a_n$. Now $c \not\subset x$ or $x \not\subset d$ which means that there is some α in $c \setminus x$ or in $x \setminus d$. The first case gives $x \in \alpha^-$, $P \subset \alpha^+$ while the second one gives $x \in \alpha^+$ and $P \subset \alpha^-$. $\square$

Lemma 4.3. *For every set S the Boolean algebra $\mathcal{P}(S)$ is a CAE.*

Proof. Let $f\colon M \to \mathcal{P}(S)$ be a map from a subset of an S_4 space X, satisfying (I). For $\alpha \in S$ define $p_\alpha\colon \mathcal{P}(S) \to \{0,1\}$ as $p_\alpha(a) = 1$ iff $\alpha \in a$. Clearly, p_α is a cp-map. By Lemma 4.1 for every $\alpha \in S$ the map $p_\alpha f$ can be extended to a cp-map $g_\alpha\colon X \to \{0,1\}$. We set

$$g(x) = \{\alpha \in S:\ g_\alpha(x) = 1\},$$

to obtain a map $g\colon X \to \mathcal{P}(S)$ which is an extension of f (an easy calculation). It remains to check that g is cp. In view of [8, Proposition I.1.12] and Lemma 4.2 this is equivalent to the fact that $g^{-1}(\alpha^+)$ is a halfspace in X for every $\alpha \in S$. Observe that $g^{-1}(\alpha^+) = \{x \in X:\ \alpha \in g(x)\} = g_\alpha^{-1}(1)$, hence $g^{-1}(\alpha^+)$ is a halfspace in X since g_α is cp. This completes the proof. $\square$

Now we immediately obtain the main theorem of this section. One can look at this as a generalization of Sikorski's theorem on injectivity of complete Boolean algebras.

Theorem 4.4. *Every complete Boolean algebra is a convexity absolute extensor.*

Proof. Let B be a complete Boolean algebra. By Stone's representation theorem there exists a set S such that B is a subalgebra of $\mathcal{P}(S)$. By the theorem of Sikorski [4] there exists a retraction $r\colon \mathcal{P}(S) \to B$. Now, let $f\colon M \to B$ be a map with the property (I) where M is a subset of an S_4 convexity space X. By Lemma 4.3 there exists a convexity preserving extension $g\colon X \to \mathcal{P}(S)$. The superposition rg is the desired extension of f. $\square$

Recall that a subset of a convexity (geometrical) space X is again a convexity (geometrical) space with the (interval) convexity generated by the traces of all convex subsets of X (see [8]). We say that a subset M of a convexity (geometrical) space X is *well-posed* in X provided $\operatorname{conv} A \cap \operatorname{conv} B = \emptyset$ whenever A, B are disjoint and convex in M, where "conv" denotes the convex hull in X. Observe that any convex subset is well-posed. Sublattices of a lattice provide examples of not necessarily convex well-posed subsets.

Recall that a convexity is binary provided each finite collection of convex sets such that any two of them intersect, has nonempty intersection.

Proposition 4.5. *If M is a well-posed subset of a convexity (geometrical) space X and Y is a binary convexity space then every cp-map $f\colon M \to Y$ has property (I).*

Proof. As the convexity of Y is binary, it is enough to show that $\operatorname{conv} f(S) \cap \operatorname{conv} f(T) \neq \emptyset$ whenever $\operatorname{conv} S \cap \operatorname{conv} T \neq \emptyset$ for $S, T \in [M]^{<\omega}$. If f is cp then, as M is well-posed, there exists an $m \in \operatorname{conv}_M S \cap \operatorname{conv}_M T$ and $f(m) \in \operatorname{conv} f(S) \cap \operatorname{conv} f(T)$. Hence $\operatorname{conv} f(S) \cap \operatorname{conv} f(T) \neq \emptyset$. $\square$

Using Theorem 4.4 and Proposition 4.5 we obtain a "geometrical" extension theorem, which is similar to Tietze-Urysohn's Extension Theorem in topology.

Theorem 4.6. *If B is a complete Boolean algebra and M is a well-posed subset of an S_4 convexity (geometrical) space X then every convexity preserving map $f\colon M \to B$ can be extended to a convexity preserving map $g\colon X \to B$.*

It is well-known that a Cantor cube $\{0,1\}^\kappa$ is a topological absolute extensor in the class of zero-dimensional spaces. On the other hand, $\{0,1\}^\kappa$ can be regarded as an algebra of subsets of κ. These two structures in a Cantor cube bring about that we might expect a topological version of Theorem 4.6. Slightly modifying the proofs of Lemmas 4.1 and 4.3 we obtain the following

Theorem 4.7. *Let X be a topological space being simultaneously a convexity space. Let M be a subset of X such that any two disjoint closed and convex in M subsets of M can be separated by a clopen halfspace in X. Then every continuous cp-map $f\colon M \to \{0,1\}^\kappa$ can be extended to a continuous cp-map $g\colon X \to \{0,1\}^\kappa$.*

It is natural to ask whether a complete Boolean algebra is the only candidate for a convexity absolute extensor. The answer is affirmative, if we consider the class of S_4 binary convexity spaces. It is known [8, Proposition II.1.15] that such spaces are of arity two.

Lemma 4.8. *Every S_4 convexity space is isomorphic to a well-posed subset of $\mathcal{P}(\mathcal{H})$ for a sufficiently large set $\mathcal{H}$.*

Proof. Let X be an S_4 convexity space, let $\mathcal{H}$ be the collection of all (proper) halfspaces in X. Define a map $\phi\colon X \to \mathcal{P}(\mathcal{H})$ by setting $\phi(x) = \{H \in \mathcal{H} :\ x \in H\}$. It is easy to show (see [8, Lemma I.3.16]) that ϕ is a convexity preserving embedding. Assume A, B are nonempty, disjoint and convex in X. There exists a halfspace $H \in \mathcal{H}$ with $A \subset H$ and $B \subset X \setminus H$. Now $\phi(A) \subset H^+$ and $\phi(B) \subset \mathcal{P}(\mathcal{H}) \setminus H^+$. This shows that $\phi(A), \phi(B)$ are separated by a halfspace, consequently $\phi(X)$ is well-posed in $\mathcal{P}(\mathcal{H})$. $\square$

Theorem 4.9. *If an S_4 binary convexity space is a convexity absolute extensor then it is isomorphic (in the category of convexity spaces) to a complete Boolean algebra.*

Proof. Let X be an S_4 space with binary convexity. By Lemma 4.8 we may assume that X is a well posed subset of an algebra $B = \mathcal{P}(\mathcal{H})$. Now if X is a CAE then the identity map $i\colon X \to X$ can be extended to a cp-retraction $r\colon B \to X$. Thus X is a retract of B. Since B is homogenous (being isomorphic to a power of a homogenous 2-element space $\{0,1\}$) we can assume that $\emptyset \in X$. Now $r(\emptyset) = \emptyset$ and consequently r is a lattice homomorphism from B into a relative subalgebra $\mathcal{P}(r(\mathcal{H}))$ of B. Hence X is a complete Boolean algebra. $\square$

We do not know whether there exists an S_3 convexity absolute extensor space which is not a complete Boolean algebra.

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