A CHARACTERIZATION OF COMPLETE BOOLEAN ALGEBRAS

WIESŁAW KUBIŚ

Abstract. Every lattice and, in particular, every Boolean algebra is a convexity space with a naturally defined convexity structure. We characterize complete Boolean algebras as the only S_3 convexity spaces having an extension property for certain classes of convexity preserving maps. This answers our question posed in [1]. Our characterization provides also a short proof of Sikorski's Extension Theorem for homomorphisms of Boolean algebras.

1. Introduction

By *a convexity* on a set X we mean, as in [7], a collection $\mathcal{G} \subset \mathcal{P}(X)$ containing $\emptyset, X$, closed under arbitrary intersections and closed under the unions of chains. The elements of $\mathcal{G}$ are called *convex sets. The convex hull* of a set $A \subset X$ is $\operatorname{conv} A = \bigcap\{G \in \mathcal{G}:\ A \subset G\}$. The convex hull of $\{a, b\}$ is called *the segment joining* a, b and denoted by $[a, b]$. The pair $(X, \mathcal{G})$ is called *a convexity space.* A convexity space X is S_4 provided for each two disjoint convex sets $A, B \subset X$ there exists *a halfspace* (i.e. a convex set with the convex complement) $H \subset X$ such that $A \subset H$ and $B \subset X \setminus H$. A convexity space is S_3 provided all one-point subsets are convex and every convex set is an intersection of halfspaces (this differs from the definition of S_3 in [7], where singletons are not presumed to be convex). A convexity space is called *binary* (or *its Helly number is at most two*) if every finite linked (i.e. meeting two by two) collection of its convex sets has nonempty intersection. This is equivalent to the condition $[a, b] \cap [a, c] \cap [b, c] \neq \emptyset$ for every $a, b, c \in X$, see [7, p. 167]. A map of convexity spaces $f\colon X \to Y$ is called *convexity preserving* (*cp* for short) provided $f^{-1}(G)$ is convex in X whenever G is convex in Y. Equivalently: $f(\operatorname{conv} S) \subset \operatorname{conv} f(S)$ for every finite $S \subset X$, see e.g. [7, p. 15]. For a general theory of convexity we refer to [7] or [3].

Our fundamental examples of convexity spaces will be lattices and, in particular, Boolean algebras. Namely, if L is a lattice then the collection of all its order-convex sublattices forms a convexity on L, which will be referred to as the natural convexity on a lattice (see [5, 6]). Observe that $[a, b] = \{x \in L:\ a \wedge b \leqslant x \leqslant a \vee b\}$ and $\operatorname{conv} S = [\inf S, \sup S]$ for a finite set S. A subset $G \subset L$ is convex iff for every $a, b \in G$, $[a, b] \subset G$. In particular all ideals and filters are convex. Every lattice is binary, see [6]. A lattice is S_4 iff it is distributive, see [5, 6]. Every lattice homomorphism is convexity preserving. Conversely,

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a cp map of lattices $f\colon K \to L$ is a homomorphism if $f(0_K) = 0_L$ or $f(1_K) = 1_L$ or f is order-preserving, see [6].

A result in [1] (Theorem 2.3 below) says that certain maps defined on subsets of S_4 convexity spaces and with values in a complete Boolean algebra can be extended to convexity preserving maps onto the whole space. The proof used the theorem of Sikorski [2] on injectivity of complete Boolean algebras. Here we give a straightforward proof, obtaining the theorem of Sikorski as a corollary. The mentioned extension theorem implies in particular that every complete Boolean algebra $\mathbb{B}$ has the following property: for every S_4 convexity space X, every cp map $f\colon G \to \mathbb{B}$ defined on a convex subset of X, can be extended to a cp map $\overline{f}\colon X \to \mathbb{B}$. We shall say that a convexity space Y is *a convexity absolute extensor* if it has the above extension property. In [1] we asked whether complete Boolean algebras are the only S_3 convexity spaces satisfying the assertion of Theorem 2.3 below. Here we prove that every S_3 convexity absolute extensor is isomorphic to a complete Boolean algebra, thus giving a positive answer. These results together provide an external characterization of complete Boolean algebras in the category of S_3 convexity spaces.

2. Extension theorem

We start with two auxiliary lemmas.

Lemma 2.1. *In every Boolean algebra, the following equivalence holds:*

$$\operatorname{conv}(A \cup B) \cap \operatorname{conv}(C \cup D) \neq \emptyset \iff \operatorname{conv}(A \cup \neg D) \cap \operatorname{conv}(C \cup \neg B) \neq \emptyset,$$

where $\neg S = \{\neg s : \ s \in S\}$ and $\neg s$ denotes the complement of s.

Proof. Suppose that $\operatorname{conv}(A \cup \neg D) \cap \operatorname{conv}(C \cup \neg B) = \emptyset$ and let H be such a halfspace that $A \cup \neg D \subset H$ and $(C \cup \neg B) \cap H = \emptyset$. Then $D \cap H = \emptyset$ and $B \subset H$. It follows that H separates $\operatorname{conv}(A \cup B)$ from $\operatorname{conv}(C \cup D)$. $\square$

Lemma 2.2. *Every linked collection of segments in a complete Boolean algebra has nonempty intersection.*

Proof. Let $\{[a_\alpha, b_\alpha]\}_{\alpha<\lambda}$ be linked. We can assume that $a_\alpha \leqslant b_\alpha$. Now $[a_\alpha, b_\alpha] \cap [a_\beta, b_\beta] \neq \emptyset$ implies $a_\alpha \leqslant b_\beta$. Setting $x = \sup_{\alpha<\lambda} a_\alpha$ we get $x \in \bigcap_{\alpha<\lambda}[a_\alpha, b_\alpha]$. $\square$

Theorem 2.3. *Let $\mathbb{B}$ be a complete Boolean algebra and let X be an S_4-convexity space. If $M \subset X$ then every map $f\colon M \to \mathbb{B}$ satisfying the condition*

$$\text{(I)} \qquad \forall\, S, T \in [M]^{<\omega} \Big(\operatorname{conv} S \cap \operatorname{conv} T \neq \emptyset \implies \operatorname{conv} f(S) \cap \operatorname{conv} f(T) \neq \emptyset\Big),$$

can be extended to a convexity preserving map $\overline{f}\colon X \to \mathbb{B}$.

Proof. Observe that the union of a chain of maps satisfying condition (I) also satisfies (I) and every map satisfying (I) is convexity preserving. Thus, it is enough to show that for a fixed $x \in X \setminus M$ there exists a map $g\colon M \cup \{x\} \to \mathbb{B}$ satisfying condition (I) and extending f. Consider the collection of intervals

$$\mathcal{A} = \{\operatorname{conv}(f(S) \cup \neg f(T)) : \ S, T \in [M]^{<\omega}, \ \operatorname{conv} S \cap \operatorname{conv}(T \cup \{x\}) \neq \emptyset\}.$$

Let $S_i, T_i \in [M]^{<\omega}$ be such that $\operatorname{conv} S_i \cap \operatorname{conv}(T_i \cup \{x\}) \neq \emptyset$, where $i = 0, 1$. Observe that $\operatorname{conv}(S_0 \cup T_1) \cap \operatorname{conv}(S_1 \cup T_0) \neq \emptyset$. Indeed, otherwise by S_4 there exists a halfspace $H \subset X$ with $S_1 \cup T_0 \subset H$ and $S_0 \cup T_1 \subset X \setminus H$. Consequently, if e.g. $x \in H$ then $\operatorname{conv} S_0 \cap \operatorname{conv}(T_0 \cup \{x\}) = \emptyset$, a contradiction. Now, condition (I) gives

$$\operatorname{conv}(f(S_0) \cup f(T_1)) \cap \operatorname{conv}(f(S_1) \cup f(T_0)) \neq \emptyset.$$

Applying Lemma 2.1 we get

$$\operatorname{conv}(f(S_0) \cup \neg f(T_0)) \cap \operatorname{conv}(f(S_1) \cup \neg f(T_1)) \neq \emptyset.$$

Thus we have shown that the collection $\mathcal{A}$ is linked.

As $\mathbb{B}$ is complete, we can find a point $y \in \bigcap \mathcal{A}$. Define $g \colon M \cup \{x\} \to \mathbb{B}$ by setting $g|M = f$ and $g(x) = y$. It remains to check that g satisfies condition (I). Let $S, T \in [M]^{<\omega}$ be such that $\operatorname{conv} S \cap \operatorname{conv}(T \cup \{x\}) \neq \emptyset$. Then $y \in \operatorname{conv}(f(S) \cup \neg f(T))$ and applying Lemma 2.1 for $A = \{y\}$, $B = f(T)$, $C = f(S)$, $D = \emptyset$, we get $\operatorname{conv} g(T \cup \{x\}) \cap \operatorname{conv} g(S) \neq \emptyset$. This completes the proof. □

In the special case when the domain is a distributive lattice, it is easy to observe that every partial lattice homomorphism satisfies condition (I). Thus, applying Theorem 2.3, we obtain the classical extension theorem of Sikorski [2].

Corollary 2.4. *Let K be a sublattice of a distributive lattice L and let $\mathbb{B}$ be a complete Boolean algebra. Then every homomorphism $f \colon K \to \mathbb{B}$ can be extended to a homomorphism $\overline{f} \colon L \to \mathbb{B}$.*

3. Convexity absolute extensors

We shall use the following characterization of Boolean algebras, which is an immediate consequence of [6, Thm. 3.5].

Lemma 3.1. *A convexity space Y is isomorphic to a Boolean algebra iff it is S_3, binary and complemented, i.e. for every $a \in Y$ there exists $b \in Y$ with $[a, b] = Y$.*

We shall also use the fact that every S_3-space is *inner transitive* [4], i.e. it satisfies the formula $(\forall\, a, b, c, d)\ d \in [a, b] \wedge c \in [a, d] \implies d \in [c, b]$. Indeed, $d \notin [c, b]$ would imply that there is a halfspace H with $d \notin H \supset [c, b]$ and then either $d \notin [a, b]$ or $c \notin [a, d]$.

Theorem 3.2. *Every S_3 convexity absolute extensor is isomorphic to a complete Boolean algebra.*

Proof. Let Y be an S_3 convexity absolute extensor. We first check the assumptions of Lemma 3.1.

Fix $a \in Y$ and consider a space $P = Y \cup \{p\}$, where $p \notin Y$, with the convexity $\mathcal{G} = \{A \subset P :\ \text{either } |A \cap \{a, p\}| = 1 \text{ or } A = P\}$. It is easy to check that $(P, \mathcal{G})$ is S_4 and $Y \in \mathcal{G}$. Let $f \colon Y \to Y$ be the identity map. Then f is cp. Let $\overline{f} \colon P \to Y$ be a cp extension of f. Since $Y \subset [a, p]$ in P we get $Y \subset [a, \overline{f}(p)]$ in Y. Thus $\overline{f}(p)$ is a complement of a.

We check that Y is binary. Fix $a, b, c \in Y$. Consider a subspace $Q = G \cup \{q\}$ of $\mathbb{R} \times \mathbb{R}$ with the lattice convexity (with coordinate-wise order), where $G = \{(0, 0), (2, 0), (1, 1), (3, 1)\}$,

$q = (4,0)$. It is easy to check that Q is S_4 and G is convex in Q. Now define $f\colon G \to Y$ by setting $f(0,0) = \neg c$, $f(2,0) = b$, $f(1,1) = \neg a$, $f(3,1) = c$, where $\neg a, \neg c$ denote the complements of a, c (which are unique by S_3). One can easily observe that f is cp. If $\overline{f}\colon Q \to Y$ is an extension of f then setting $y = \overline{f}(q)$ we get $b, c \in [\neg a, y]$ and $b \in [\neg c, y]$. Applying inner transitivity we obtain $y \in [a,b] \cap [a,c] \cap [b,c]$.

Thus, applying Lemma 3.1, we see that Y is isomorphic to a Boolean algebra. Fix a partial order $\leqslant$ on Y induced by a given isomorphism. We show that every maximal linearly ordered subset $L \subset Y$ is complete, which implies the completeness of Y itself. Consider $A, B \subset L$ such that $a < b$ for all $a \in A, b \in B$. Let $X = A \cup B \cup \{p\}$ where $p \notin L$ and define a linear order $\leqslant^*$ on X by letting

$$x \leqslant^* y \text{ iff } \begin{cases} x = p & \text{or} \\ x \in B \ \&\ y \in A & \text{or} \\ x, y \in A \ \&\ x \leqslant y & \text{or} \\ x, y \in B \ \&\ y \leqslant x. \end{cases}$$

Every linearly ordered set is an S_4 convexity space (being a distributive lattice). Define $f\colon A \cup B \to Y$ by setting $f(a) = \neg a$ for $a \in A$ and $f(b) = b$ for $b \in B$. Clearly, f is cp; if $\overline{f}\colon X \to Y$ is a cp extension of f then by inner transitivity we get $a \leqslant \overline{f}(p) \leqslant b$ for all $a \in A$, $b \in B$. Now, if B is the set of all upper bounds of A then $\overline{f}(p) = \sup A$ in L. □

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Institute of Mathematics, Silesian University, ul. Bankowa 14, 40-007 Katowice, Poland

(Instytut Matematyki, Uniwersytet lski)

E-mail address: kubis@ux2.math.us.edu.pl