Exercises for Mathematical Logic (16 Nov 2022)

15. For any formula $\varphi(x)$ and variable y free for x in φ, show that the formula $\exists y\,(\exists x\,\varphi(x) \to \varphi(y))$ is provable.

16. Using Vaught's test, show the completeness of the theory of a successor: it has a language with one unary function symbol s, and axioms $s(x) = s(y) \to x = y$, $\forall x\,\exists y\,s(y) = x$, and $s^n(x) \neq x$ for each $n \in \mathbb{N}_{>0}$, where s^n denotes the n-fold iteration of s (i.e., $s^0(x)$ is x, and s^{n+1} is $s(s^n(x))$).

17. For each $n \in \mathbb{N}$, let P_n denote the path graph of length n, i.e., the structure $\langle [n], E_n \rangle$, where $[n] = \{0, \dots, n-1\}$ and $E_n = \{\langle i, j \rangle \in [n]^2 : |i - j| = 1\}$. Show that there is no sentence φ such that for all $n \in \mathbb{N}$, $P_n \vDash \varphi$ iff n is odd. [Hint: Adapt the previous exercise.]

18. Fix a field F. The theory of vector spaces over F has a language consisting of the language $\{+, -, 0\}$ of abelian groups and unary functions $a \cdot x$ for each $a \in F$; it has the usual algebraic axioms (axioms of abelian groups, $ab \cdot x = a \cdot (b \cdot x)$, $1 \cdot x = x$, $(a+b) \cdot x = a \cdot x + b \cdot x$, $a \cdot (x+y) = a \cdot x + a \cdot y$). Show that the theory of infinite vector spaces over F (i.e., with additional axioms $\exists x_0 \dots \exists x_n \bigwedge_{i<j} x_i \neq x_j$ for $n \in \mathbb{N}$) is complete and κ-categorical for all infinite $\kappa > |F|$. [Hint: Every vector space has a basis.]