Complexity of unification and admissibility with parameters in transitive modal logics

Emil Jeřábek
jerabek@math.cas.cz
http://math.cas.cz/~jerabek/

Institute of Mathematics of the Academy of Sciences, Prague

Admissibility and unification

Propositional logics

Propositional logic L:

Language: formulas built from atoms (variables) $\{x_n : n \in \omega\}$ using a fixed set of connectives of finite arity

Consequence relation: a relation $\Gamma \vdash_L \varphi$ between sets of formulas and formulas such that

- $\varphi \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\Gamma, \Delta \vdash_L \varphi$
- $\Gamma, \Delta \vdash_L \varphi$ and $\forall \psi \in \Delta\, \Gamma \vdash_L \psi$ imply $\Gamma \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every substitution σ

Algebraizable logics

A logic L is finitely algebraizable wrt a class K of algebras if there is a finite set $E(x,y)$ of formulas and a finite set $T(x)$ of equations such that

- $\Gamma \vdash_L \varphi \Leftrightarrow T(\Gamma) \vDash_K T(\varphi)$
- $\Delta \vDash_K t \approx s \Leftrightarrow E(\Delta) \vdash_L E(t,s)$
- $x \dashv\vdash_L E(T(x))$
- $x \approx y \mathrel{=\!\mid\!\models}_K T(E(x,y))$

In modal logic, we will have:
$T(x) = \{x \approx 1\}$, $E(x,y) = \{x \leftrightarrow y\}$, K is a variety of modal algebras

Equational unification

Θ: a background equational theory (or a variety of algebras)

Basic Θ-unification problem:
Given a set of equations $\Gamma = \{t_1 \approx s_1, \dots, t_n \approx s_n\}$, is there a substitution σ (a Θ-unifier of Γ) s.t.

$$\sigma(t_1) =_\Theta \sigma(s_1), \dots, \sigma(t_n) =_\Theta \sigma(s_n)?$$

If L is a logic algebraizable wrt a (quasi)variety K:

An L-unifier of a formula φ is σ such that $\vdash_L \sigma(\varphi)$

- L-unifier of φ = K-unifier of $T(\varphi)$
- K-unifier of $t \approx s$ = L-unifier of $E(t, s)$
- Sets reduce to single formulas if L has well-behaved conjunction

Admissible rules

Single-conclusion rule: $\Gamma \;/\; \varphi$ (Γ finite set of formulas)

Multiple-conclusion rule: $\Gamma \;/\; \Delta$ (Γ, Δ finite sets of formulas)

- $\Gamma \;/\; \Delta$ is L-derivable (or valid) if $\Gamma \vdash_L \delta$ for some $\delta \in \Delta$
- $\Gamma \;/\; \Delta$ is L-admissible (written as $\Gamma \mathrel{|\!\!\sim}_L \Delta$) if every L-unifier of Γ also unifies some $\delta \in \Delta$

$$T(\Gamma \;/\; \Delta) := \bigwedge_{\gamma \in \Gamma} T(\gamma) \Rightarrow \bigvee_{\delta \in \Delta} T(\delta):$$

- $\Gamma \;/\; \Delta$ is derivable iff $T(\Gamma \;/\; \Delta)$ holds in all K-algebras
- $\Gamma \;/\; \Delta$ is admissible iff $T(\Gamma \;/\; \Delta)$ holds in free K-algebras

Note: Γ is unifiable iff $\Gamma \mathrel{|\!\!\not\sim}_L \varnothing$

Parameters

In real life, propositional atoms model both "variables" and "constants"

We don't want to allow substitution for constants

Example (description logic):

(1) $\forall\mathrm{child}.(\neg\mathrm{HasSon}\sqcap\exists\mathrm{spouse}.\top)$

(2) $\forall\mathrm{child}.\forall\mathrm{child}.\neg\mathrm{Male}\sqcap\forall\mathrm{child}.\mathrm{Married}$

(3) $\forall\mathrm{child}.\forall\mathrm{child}.\neg\mathrm{Female}\sqcap\forall\mathrm{child}.\mathrm{Married}$

Good: Unify (1) with (2) by $\mathrm{HasSon}\mapsto\exists\mathrm{child}.\mathrm{Male}$, $\mathrm{Married}\mapsto\exists\mathrm{spouse}.\top$

Bad: Unify (2) with (3) by $\mathrm{Male}\mapsto\mathrm{Female}$

Admissibility with parameters

In unification theory, it is customary to consider unification with free constants

Set-up with two kinds of atoms:

- variables $\{x_n : n \in \omega\}$
- parameters (constants) $\{p_n : n \in \omega\}$

Substitutions only modify variables, we require $\sigma(p_n) = p_n$

Adapt accordingly other notions:

- L-unifier, L-admissible rule

Caveat: "Propositional logic" is always assumed to be closed under substitution for parameters

Transitive modal logics

Transitive modal logics

Normal modal logics with a single modality $\Box$, include the transitivity axiom $\Box x \to \Box\Box x$ (i.e., $L \supseteq \mathbf{K4}$)

Common examples: various combinations of

logic	axiom (on top of $\mathbf{K4}$)	finite rooted transitive frames
$\mathbf{S4}$	$\Box x \to x$	reflexive
$\mathbf{D4}$	$\Diamond\top$	final clusters reflexive
$\mathbf{GL}$	$\Box(\Box x \to x) \to \Box x$	irreflexive
$\mathbf{K4Grz}$	$\Box(\Box(x \to \Box x) \to x) \to \Box x$	no proper clusters
$\mathbf{K4.1}$	$\boxdot\Diamond x \to \diamonddot\Box x$	no proper final clusters
$\mathbf{K4.2}$	$\Diamond\boxdot x \to \Box\diamonddot x$	unique final cluster
$\mathbf{K4.3}$	$\Box(\boxdot x \to y) \lor \Box(\Box y \to x)$	linear (chain of clusters)
$\mathbf{K4B}$	$x \to \Box\Diamond x$	lone cluster
$\mathbf{S5}$	$= \mathbf{S4} \oplus \mathbf{B}$	lone reflexive cluster

Admissibility in transitive modal logics

Much is known about admissibility and unification in logics with suitable frame extension properties:

- Semantic characterization of admissible rules, decidability of admissibility (even with parameters) [Rybakov]
- Existence of projective approximations and computable finite complete sets of unifiers [Ghilardi]
- Explicit bases of admissible rules [J.]

Various results were generalized to the setting with parameters in [J13]

Complexity of admissibility and unification

Complexity of parameter-free unification and (in)admissibility [J07]:

- Logics of branching 1: usually NP-complete
- Extensible logics of infinite branching: NEXP-complete
- General logics satisfying certain weak condition: NEXP-hard

This talk: unification and (in)admissibility with parameters

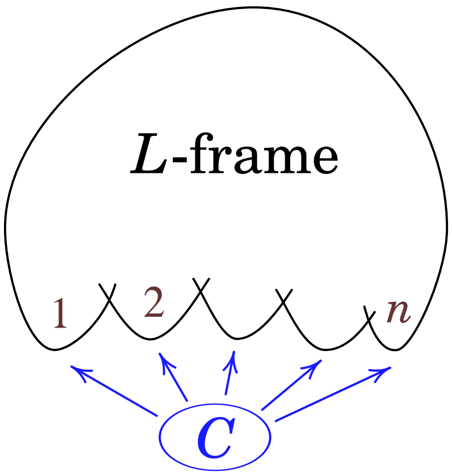
- Lower bounds for broad classes of logics
- Matching upper bounds for cluster-extensible logics
- Complexity depends on semantic properties of the logic

Cluster-extensible logics

L a transitive modal logic with fmp, $n \in \omega$, C a finite cluster type (irreflexive $\bullet$, k-element reflexive ⓚ):

A finite rooted frame F is of type $\langle C, n\rangle$ if its root cluster $\mathrm{rcl}(F)$ is of type C and has n immediate successor clusters

L is $\langle C, n\rangle$-extensible if:
For every type-$\langle C, n\rangle$ frame F, if $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame, then so is F



L is cluster-extensible (clx), if it is $\langle C, n\rangle$-extensible whenever it has some type-$\langle C, n\rangle$ frame

Properties of clx logics

Examples: All combinations of $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{GL}$, $\mathbf{D4}$, $\mathbf{K4Grz}$, $\mathbf{K4.1}$, $\mathbf{K4.3}$, $\mathbf{K4B}$, $\mathbf{S5}$, $\pm$ bounded branching ($\mathbf{K4BB}_k$) or cluster size ($\mathbf{K4BC}_k$)

Nonexamples: $\mathbf{K4.2}$, $\mathbf{S4.2}$, $\dots$

For every clx logic L:

- L is finitely axiomatizable
- L has the exponential-size model property
- L is $\forall\exists$-definable on finite frames
- L is PSPACE-complete (if branching ≥ 2) or coNP-complete

Variants of clx logics

The definition can be tweaked to cover other kinds of logics:

- Logics with a single top cluster (extensions of $\mathbf{K4.2}$)
  - Top-restricted cluster-extensible (tclx) logics: extension condition only for frames with a single top cluster
  - Examples: joins of $\mathbf{K4.2}$ with clx logics
- Superintuitionistic logics
  - Behave much like their largest modal companion (Blok–Esakia isomorphism)
  - The only (t)clx logics are $\mathbf{IPC}$, $\mathbf{T}_n$, $\mathbf{KC}$, $\mathbf{KC}+\mathbf{T}_n$ (NB: $\mathbf{T}_1 = \mathbf{LC}$, $\mathbf{T}_0 = \mathbf{CPC}$)

Tight predecessors

P a finite set of parameters, C a finite cluster type, $n \in \omega$

- Consider frames W with fixed valuation of parameters
- W is $\langle C, n\rangle$-extensible if for every $E \subseteq \mathbf{2}^P$, $0 < |E| \le |C|$, and every $X = \{w_1, \dots, w_n\} \subseteq W$, there is a tight predecessor (tp) $\{u_e : e \in E\} \subseteq W$:

$$u_e \vDash P^e, \qquad u_e{\uparrow} = X{\Uparrow} \overbrace{\cup \{u_{e'} : e' \in E\}}^{C \text{ reflexive}}$$

- If L is a clx logic, an L-frame is L-extensible if it is $\langle C, n\rangle$-extensible whenever L is
- For tclx logics: if $n > 0$, only $\{w_1, \dots, w_n\}$ below the same top cluster have tp's

Semantics for admissible rules

Theorem:
If L is a clx or tclx logic, tfae:

- $\Gamma \nvdash_L \Delta$
- $\Gamma \,/\, \Delta$ fails in some L-extensible model
- $\Gamma \,/\, \Delta$ fails in an exponential-size L-model that "approximates" an extensible model wrt subformulas of $\Gamma \cup \Delta$

Note: L-extensible models are normally infinite

Upper bound strategy

Semantic characterization $\Rightarrow$ unifiability and inadmissibility in any (t)clx logic is Σ_2^{EXP}:

$$\exists\,\text{model}\,\forall\, E \subseteq \mathbf{2}^P \ldots$$

Optimization in certain cases:

- Bounded cluster size: $\forall E \subseteq \mathbf{2}^P$ becomes a poly-size quantifier
- Width 1:
  - The model is an upside-down tree of clusters
  - An alternating TM can seach for it while keeping only one partial branch
    (≈ the usual proof that $\vdash_L$ is in PSPACE)

Lower bound conditions

Main principle: Hardness of L-unifiability stems from finite configurations that occur as subframes in L-frames

I.e., if there are subreductions from some general L-frames to a particular finite frame or a sequence of frames, L-unifiability is $\mathcal{C}$-hard.

Example conditions:

- L has unbounded depth:
  L-frames subreduce to arbitrarily long finite chains
- L has unbounded cluster size:
  L-frames subreduce to arbitrarily large finite clusters
- L has width ≥ 2:
  an L-frame subreduces to a 3-element fork

Lower bound strategy

Reduce to L-unifiability a $\mathcal{C}$-complete problem, e.g.:

- PSPACE: validity of quantified Boolean sentences
- $\Sigma_k^{\mathrm{EXP}}/\Pi_k^{\mathrm{EXP}}$: validity of Σ_k^2/Π_k^2-sentences on finite sets

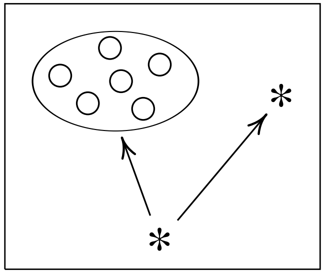
$$\exists X_1 \subseteq \mathcal{P}([n])\,\forall X_2 \subseteq \mathcal{P}([n])\,\exists t_1,\dots,t_c \subseteq [n]\,\varphi(i \in t_\alpha, t_\alpha \in X_j, \dots)$$

- generally: $\exists$ simulated by variables, $\forall$ by parameters
- $\forall X \subseteq \mathcal{P}([n])$: parameter assignments realized in a cluster
- $\exists X \subseteq \mathcal{P}([n])$: single variable x
  - use antichains to enforce consistency:
  - $w \vDash \sigma(x)$ unaffected by a change of parameters in points $v \not\geq w$

Σ_2^{EXP} bounds

Recall: $\Sigma_2^{\mathrm{EXP}} = \mathrm{NEXP}^{\mathrm{NP}}$

Lower bound:
L-unifiability is Σ_2^{EXP}-hard if $\forall n$ an L-frame subreduces to a rooted frame containing an n-element cluster and an incomparable point.



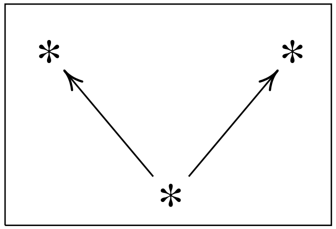
Upper bound:
If L is a clx or tclx logic, then L-inadmissibility is in Σ_2^{EXP}.

Examples: $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S4.1}$, $\mathbf{S4.2}$, ... (± bounded branching)

NEXP **bounds**

Lower bound:

If L has width ≥ 2, then L-unifiability is NEXP-hard.



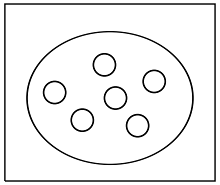
Upper bound:

If L is a clx or tclx logic of bounded cluster size, then L-inadmissibility is in NEXP.

Examples: **GL**, **K4Grz**, **S4Grz**, **S4Grz.2**, **IPC**, **KC**, ...
($\pm$ bounded branching)

coNEXP **bounds**

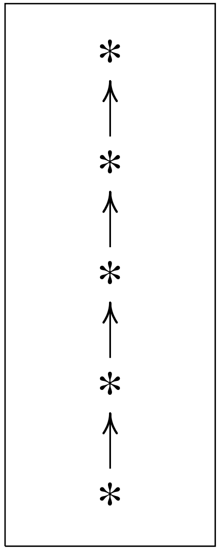
Lower bound:
If L has unbounded cluster size, then L-unifiability is coNEXP-hard.



Upper bound:
If L is a clx logic of width 1, then L-inadmissibility is in coNEXP.

Examples: $\mathbf{S5}$, $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\ldots$

PSPACE **and below**



Lower bound:
L-unifiability is PSPACE-hard, unless L is a tabular logic of width 1.

Upper bound:
If L is a clx logic of width 1 and bounded cluster size, then L-admissibility is in PSPACE.

Examples: $\mathbf{GL.3}$, $\mathbf{K4Grz.3}$, $\mathbf{S4Grz.3}$, $\mathbf{LC}$, ...

Remaining cases:
If L is a tabular logic of width 1 and depth d, then L-unification and L-inadmissibility are Π^{P}_{2d}-complete.

Examples: $\mathbf{CPC}$, $\mathbf{G}_{d+1}$, $\mathbf{S5} \oplus \mathbf{Alt}_k$, $\mathbf{K4} \oplus \Box\bot$, ...

Complexity summary for clx logics

We get the following classification for clx logics:

<table>
<tr><th colspan="2">logic</th><th rowspan="2">$\nvdash_L$</th><th colspan="2">$\not\mathrel{|\!\sim}_L$</th><th rowspan="2">examples</th></tr>
<tr><th>cluster size</th><th>bran-ching</th><th>par.-free</th><th>with param's</th></tr>
<tr><td rowspan="2">$< \infty$</td><td>0</td><td colspan="2" rowspan="3">NP-complete</td><td>Π^{P}_2-c.</td><td>$\mathbf{S5} \oplus \mathbf{Alt}_k$, $\mathbf{CPC}$</td></tr>
<tr><td>1</td><td>PSPACE-c.</td><td>$\mathbf{GL.3}$, $\mathbf{LC}$</td></tr>
<tr><td>∞</td><td>≤ 1</td><td>coNEXP-c.</td><td>$\mathbf{S5}$, $\mathbf{S4.3}$</td></tr>
<tr><td>$< \infty$</td><td rowspan="2">≥ 2</td><td rowspan="2">PSPACE-c.</td><td colspan="2">NEXP-complete</td><td>$\mathbf{GL}$, $\mathbf{S4Grz}$, $\mathbf{IPC}$</td></tr>
<tr><td>∞</td><td></td><td>Σ^{EXP}_2-c.</td><td>$\mathbf{K4}$, $\mathbf{S4}$</td></tr>
</table>

With parameters, unifiability and inadmissibility have the same complexity

Θ_2^{EXP} bounds

For tclx logics, we need one more exotic class
Θ_2^{EXP} is the exponential version of Θ_2^{P}:

$$\Theta_2^{\mathrm{EXP}} := \mathrm{EXP}^{\mathrm{NP[poly]}} = \mathrm{EXP}^{\|\mathrm{NP}} = \mathrm{P}^{\mathrm{NEXP}} = \mathrm{PSPACE}^{\mathrm{NEXP}}$$

Lower bound:
L-unifiability is Θ_2^{EXP}-hard if $\forall n$ there is a connected L-frame (in mathematical sense) of cluster size $\geq n$ and width ≥ 2.

Upper bound:
If L is a tclx logic of bounded inner cluster size, then L-admissibility is in Θ_2^{EXP}.

Example: $\mathbf{S4.2} \oplus \mathbf{S4.1.4}$

Complexity summary for tclx logics

Classification for tclx logics:
(NB: they extend $\mathbf{K4.2}$ and have branching ≥ 2 by definition)

<table>
<tr><th colspan="2">logic</th><th rowspan="2">$\nvdash_L$</th><th colspan="2">$\not\mathrel{|\!\sim}_L$</th><th rowspan="2">examples</th></tr>
<tr><th>inner cl. size</th><th>top cl. size</th><th>par.-free</th><th>w/ param's</th></tr>
<tr><td rowspan="2">$< \infty$</td><td>$< \infty$</td><td rowspan="3">PSPACE-c.</td><td rowspan="3" colspan="1">NEXP-complete</td><td>NEXP-complete</td><td>$\mathbf{GL.2}$, $\mathbf{Grz.2}$, $\mathbf{KC}$</td></tr>
<tr><td>∞</td><td>Θ_2^{EXP}-c.</td><td>$\mathbf{S4.1.4} \oplus \mathbf{S4.2}$</td></tr>
<tr><td>∞</td><td></td><td>Σ_2^{EXP}-c.</td><td>$\mathbf{K4.2}$, $\mathbf{S4.2}$</td></tr>
</table>

Again: with parameters, unifiability and inadmissibility have the same complexity

Hereditary hardness

Can we fully classify the complexity of unifiability for all transitive logics L?

- Hopeless as such: e.g., $\vdash_L$ can be undecidable with arbitrary Turing degree
- But: we can determine the minimal complexity of unifiability among the sublogics of L

Definition: Unifiability has hereditary hardness $\mathcal{C}$ below L if

- L'-unifiability is $\mathcal{C}$-hard for all $L' \subseteq L$
- L'-unifiability is $\mathcal{C}$-complete for some $L' \subseteq L$

Hereditary hardness (cont'd)

Theorem:
The hereditary hardness of unifiability below any transitive logic is one of Σ_2^{EXP}, Θ_2^{EXP}, EXPBH_2, NEXP, coNEXP, PSPACE, or Π_{2d}^{P}.

Here, a language is in EXPBH_2 if it can be written as the difference of two NEXP languages

Boolean hierarchy over NEXP:

$$\mathrm{EXPBH}_1 = \mathrm{NEXP}$$

$$\mathrm{EXPBH}_{k+1} = \{A \smallsetminus B : A \in \mathrm{NEXP}, B \in \mathrm{EXPBH}_k\}$$

Example: L-unifiability (and L-inadmissibility) is EXPBH_2-complete for $L = \mathbf{S5} \cap \mathbf{S4Grz}$

Unifiability vs. inadmissibility

- Parameter-free unifiability is often much easier than inadmissibility: e.g., extensions of $\mathbf{IPC}$, $\mathbf{D4}$, $\mathbf{GL}$
- With parameters, unifiability has the same complexity as inadmissibility for all clx and tclx logics
- However, this is not a general principle

Example: $L = \mathbf{GL} \cap \mathbf{S4.3}$

- L-frames are disjoint sums of $\mathbf{GL}$-frames and $\mathbf{S4.3}$-frames
- L-unifiability is EXPBH_2-complete
- single-conclusion L-inadmissibility is EXPBH_4-complete
- multiple-conc. L-inadmissibility is $\mathrm{EXP}^{\mathrm{NP}[\log n]}$-complete

Thank you for attention!

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