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TOEPLITZ OPERATORS ON BERGMAN-TYPE SPACES

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A SURVEY OF THE PRESENT STATE OF THE SUBJECT

The subject of this thesis are Toeplitz operators on Bergman-type spaces[1], an area which has received some attention during the last decade. These operators arise mainly in two areas of mathematics: in connection with quantum mechanics (the quantization procedures on symmetric spaces, cf. [5], [6]) and in the theory of subnormal operators (the basic model for a subnormal operator is the Bergman shift; see [9], [23]). Whereas the theory of these operators on the Hardy space H^2 is by now well understood and has become classical (cf. e.g. [22] and the bibliography given therein), its area measure counterparts are far less tractable, and the results are sparser and often markedly different (see, for instance, [2] or [6]). It is natural to ask if this is not because the class of Toeplitz operators on a Bergman-type space is, in some sense, larger than on H^2. The classical Toeplitz operators are characterized by the intertwining relation $S^*TS = T$ (see f.i. [24]), and so form a w^*-closed subset of $\mathcal{B}(H^2)$ of infinite codimension. In my paper [16], I have shown that such a characterization is impossible on $A^2(\mathbf{D})$. In [17], it turned out that Toeplitz operators on $A^2(\mathbf{D})$ are, in fact, dense in $\mathcal{B}(A^2(\mathbf{D}))$ in the strong operator topology, which gives an affirmative answer to the conjecture above. Further progress has been attempted by Gautrin [13], who proved even norm-density; unfortunately, his proof contained an error, and so the problem of norm-density of Toeplitz operators remained open. So did also the similar problem whether the C^*-algebra generated by them contains all bounded operators (and if not, what is it?). The corresponding C^*-algebra on the Hardy space is closely related to the theory of Toeplitz (or symbol) calculus (cf. [22], Appendix 4), the study of which has contributed to the development of powerful techniques of operator theory (dilation theory, commutant lifting theorem). In case the corresponding C^*-algebra on the Bergman space contained all bounded operators, an analogue of such a calculus could be of exceptional interest. Some attempts in this direction have been made e.g. by McDonald and Sundberg (for symbols in a certain C^*-subalgebra of $L^\infty(\mathbf{D})$; see [2] and the references therein). A closely related question of compactness of Hankel operators on Bergman-type spaces has been intensively studied by various authors — Axler et al. [3], Zhu, Stroethoff, Zheng, Berger and Coburn (see [6], [27]); these issues, however, are not discussed in this thesis.

[1] See the end of Chapter 1 for terminology and notation.

THE OBJECTIVE AND METHODS OF THIS THESIS

The objective of this thesis is a solution to some of the problems mentioned above. It is proved that the norm closure of the set of all Toeplitz operators on a Bergman-type space contains all compact operators (Chapter 2); it does not, however, contain all bounded ones, and neither does even the C^*-algebra generated by this set (Chapter 3). An unexpected fact is that these C^*-algebras are always contained in certain C^*-algebra $\mathcal{A}$ which does not depend on the underlying space (see Chapter 3 for a more precise statement). An attempt to construct a Toeplitz calculus on Bergman-type spaces, alluded to above, can be found in Chapter 4; it seems that the role of the Nagy-Foias dilation could be played by certain "dilating" to the Calkin algebra. The Berezin transform, which plays an important role in the questions concerning compactness of Toeplitz operators (cf. [6], [2], [27]) seems to be important also from the viewpoint of the theory developed in Chapter 4; therefore the last Chapter 5 is devoted to its more detailed study.

Most methods are functional-analytic, or belong to operator theory and complex function theory. Rudiments of interpolation theory, potential theory and Riemannian geometry are needed in Chapter 5; algebraic topology (cohomotopy groups) appears in Remark 3.26.

RESULTS OF THE THESIS

Chapter 1 contains some preparatory material; most results are more or less known, although frequently their proofs cannot be found in the literature ("folk theorems"), so they are assembled here. The items 1.12, 1.15, 1.18 and 1.19 are mine, and so are all presented proofs.

In Chapter 2, density of Toeplitz operators on Bergman-type spaces is considered. It is proved that they are SOT-dense in the set of all bounded operators and that their norm closure contains all compact operators. A slightly simplified version of my results from [16] is reproduced here, together with a new proof and a generalization to arbitrary domains $\Omega \subset \mathbf{C}^N$.

In Chapter 3, it is proved that the C^*-algebra generated by the Toeplitz operators does not contain all bounded linear operators. An unexpected fact is that this algebra is always contained in a certain C^*-algebra which is the same (more precisely: spatially C^*-isomorphic) for $A^2(\mathbf{D})$, $A^2(\mathbf{C})$, H^2, $A^2(\Omega)$ for a wide class of domains $\Omega \subset \mathbf{C}$, and only slightly different for $A^2(\mathbf{C}^N)$ (the difference seems to be of topological nature and related to the dimension, cf. Remark 3.26). The items 3.11, 3.13, 3.14 and 3.16 generalize a number of assertions from [3], by which they were inspired. A related class of spaces $H^2(\rho)$ is also discussed.

In Chapter 4, an attempt is made to construct an analogue of the classical Toeplitz (or symbol) calculus on Bergman-type spaces (see the beginning of the chapter for a more explicit discussion). The problem is rather difficult, and only a partial solution is presented. Lemma 4.1 follows from the results of [3] or [23]; a very short and elegant proof is presented here, based on our Theorem 2.4. In 4.3 — 4.6, the techniques due to Calkin [10] are developed; 4.12 mentions some classical results, while 4.16 was inspired by Douglas ([12], 7.47 ff.). The proof of Lemma 4.19 is reproduced here from [6] (slightly simplified) for the purpose of subsequent discussion.

Some of the results from Chapters 1 and 4, as well as results of other authors ([5], [6], some preprints of Zhu, Stroethoff, Zheng, etc.) from adjacent areas, indicate that a fundamental role in the theory is played by the Berezin transform B; for this reason, the last chapter is devoted to its study. The operator B is shown to be bounded on various L^p spaces and to be related to the Laplace or the Laplace-Beltrami operator, respectively. For some classes of functions f, conditions for f to be invariant under B are given, and the iterates $B^n f$ are shown to converge. The items 5.1 – 5.4 and 5.16 are more or less known ("folk theorems"), and so is perhaps 5.29; 5.18 and 5.28 are classical; 5.12 is a generalization of the Schur test, and 5.25 may be compared with a theorem in [19] (see Remark 5.26).

Unless stated otherwise, all the results to follow belong to the present author. Proofs are usually supplied for the "folk theorems" mentioned above. When other authors' results are needed, the references are always given in the text and, except for 4.3 – 4.6 and 4.19, the proofs are not reproduced.

Chapter 1. INTRODUCTION

The purpose of this chapter is to set up notation and to introduce some basic definitions. The style is brief; some proofs are omitted and can be found e.g. in [2] or [6]. The symbol $\mathbf{C}$ will allways denote the complex plane, $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ the open unit disc, $\mathbf{T}$ its boundary, the unit circle; $\mathbf{Z}$ is the set of all integers, and $\mathbf{N}$ the set of all nonnegative ones.

Convention: By a domain in $\mathbf{C}^N$ we mean an open, connected, nonempty and *proper* subset Ω of $\mathbf{C}^N$ (written: $\Omega \subset \mathbf{C}^N$), for some integer $N \geq 1$. Thus, $\mathbf{C}^N$ itself is *not* considered to be a domain. We shall write $\Omega \subseteq \mathbf{C}^N$ to express that Ω is either a domain in $\mathbf{C}^N$ in the above sense, or $\Omega = \mathbf{C}^N$.

The symbol dz denotes the Lebesgue measure in $\mathbf{C}^N$, for all $N \geq 1$, while $L^p(\Omega, d\rho)$ stands for the usual Lebesgue space on Ω with respect to a measure ρ; if ρ is omitted, the Lebesgue measure is understood.

If Ω is a domain in $\mathbf{C}^N$ and $1 \leq p \leq \infty$, we may define

$$A^p(\Omega) := \{f \in L^p(\Omega, dz) : f \text{ is analytic on } \Omega\}.$$

This is a closed subspace of $L^p(\Omega, dz)$. For $p = \infty$, $A^\infty(\Omega) = H^\infty(\Omega)$, the space of all bounded analytic functions on Ω. For $p = 2$, $A^2(\Omega)$ becomes a Hilbert space (with the inner product from L^2). The spaces $A^2(\Omega)$ will be termed *Bergman spaces*; the space $A^2(\mathbf{D})$, which is of particular interest, will be referred to as *the* Bergman space. The latter will usually be considered as a subspace not of $L^2(\mathbf{D}, dz)$, but of $L^2(\mathbf{D}, d\nu)$, with the norm and scalar product modified accordingly; here

$$d\nu(z) = \frac{1}{\pi}\, dz$$

is a multiple of the Lebesgue measure chosen so that $\mathbf{D}$ had measure 1. If $f = \sum_{n=0}^{\infty} f_n z^n$ is holomorpic[2] on $\mathbf{D}$, a simple calculation shows that

$$\int_{\mathbf{D}} |f(x)|^2 \, d\nu(x) = \sum_{n=0}^{\infty} \frac{|f_n|^2}{n+1}.$$

Consequently, $f \in A^2(\mathbf{D})$ if and only if the last expression is finite. The scalar product of f and $g = \sum_{n=0}^{\infty} g_n z^n$, $f, g \in A^2(\mathbf{D})$, is given by

$$\langle f, g \rangle_{A^2(\mathbf{D})} = \sum_{n=0}^{\infty} \frac{f_n \overline{g_n}}{n+1}.$$

The set

$$\{\sqrt{n+1}\, z^n\}_{n \in \mathbf{N}} \tag{1}$$

[2] a synonym for *analytic* throughout this paper

is an orthonormal basis for $A^2(\mathbf{D})$. The polynomials are dense in $A^2(\mathbf{D})$.

(Observe that the last sentence, in general, need not be valid for $A^2(\Omega)$. If we take $\Omega = \mathbf{D} \setminus \langle 0, 1)$, then the closure of the polynomials in $A^2(\Omega)$ is precisely $A^2(\mathbf{D}) \subset A^2(\Omega)$. The Riemann mapping function $\Phi : \Omega \to \mathbf{D}$ belongs to $A^2(\Omega)$ – even to $A^\infty(\Omega)$ – but not to $A^2(\mathbf{D})$.)

The space $A^2(\mathbf{D})$ is a reproducing kernel space in the sense of Aronszajn [1]. Denote P_+ the orthogonal projection of $L^2(\Omega)$ onto $A^2(\Omega)$.

Proposition 1.1. *For each $\lambda \in \Omega \subset \mathbf{C}^N$, the linear functional $f \mapsto f(\lambda)$ on $A^2(\Omega)$ is bounded; consequently, $f(\lambda) = \langle f, g_\lambda \rangle >$ for some $g_\lambda \in A^2(\Omega)$. Further,*

$$\|g_\lambda\| \le \frac{1}{(\gamma_N R^{2N})^{1/2}},$$

where γ_N is the (Lebesgue) volume of the unit ball of $\mathbf{C}^N$ and $R = \operatorname{dist}(\lambda, \mathbf{C}^N \setminus \Omega)$.

Proof. Let R be the distance of λ to $\mathbf{C}^N \setminus \Omega$ and

$$F_\lambda(z) = \begin{cases} 0 & \text{if } |z-\lambda| \ge R, \\ (\gamma_N R^{2N})^{-1} & \text{if } |z-\lambda| < R. \end{cases}$$

Then

$$\int_\Omega |F_\lambda(z)|^2 \, dz = \gamma_N R^{2N} \left(\frac{1}{\gamma_N R^{2N}} \right)^2 = \frac{1}{\gamma_N R^{2N}} < +\infty,$$

so $F_\lambda \in L^2(\Omega)$. Hence, for arbitrary $f \in A^2(\Omega)$,

$$\langle f, P_+ F_\lambda \rangle_{A^2(\Omega)} = \langle f, F_\lambda \rangle_{L^2(\Omega)} = \frac{1}{\gamma_N R^{2N}} \int_{|z-\lambda|<R} f(z) \, dz,$$

which equals $f(\lambda)$ by the mean value theorem; hence, we may take $g_\lambda = P_+ F_\lambda$. Finally,

$$\|g_\lambda\|_2^2 \le \|F_\lambda\|_2^2 = \frac{1}{\gamma_N R^{2N}}.$$

□

For $\Omega = \mathbf{D}$, the reproducing kernel may be written down explicitly – namely,

$$g_\lambda(z) = \frac{1}{(1-\overline{\lambda} z)^2} = \sum_{n=0}^\infty (n+1) \overline{\lambda}^n z^n.$$

Let Ω be a domain in $\mathbf{C}^N$ and $\phi \in L^\infty(\Omega)$. We define the multiplication operator $M_\phi : L^2(\Omega) \to L^2(\Omega)$, the Toeplitz operator $T_\phi : A^2(\Omega) \to A^2(\Omega)$ and the Hankel operator $H_\phi : A^2(\Omega) \to L^2(\Omega) \ominus A^2(\Omega)$ with symbol ϕ, respectively, by the formulas

$$M_\phi f = \phi f, \qquad T_\phi f = P_+ M_\phi f, \qquad H_\phi f = (I - P_+) M_\phi f.$$

These operators are clearly bounded, their norms not exceeding $\|\phi\|_\infty$. If $\phi \in A^\infty(\Omega)$, H_ϕ is zero, while $T_\phi = M_\phi \restriction A^2(\Omega)$. The mappings $\phi \mapsto M_\phi$, $\phi \mapsto T_\phi$, $\phi \mapsto H_\phi$ are linear. The formulas below are well-known in the theory of Toeplitz and Hankel operators on the Hardy space H^2; they remain in force in the present setting.

Proposition 1.2. *For arbitrary $f, g \in L^\infty(\Omega)$,*

$$[T_f, T_g) := T_{fg} - T_f T_g = H^*_{\overline{f}} H_g,$$

$$[T_f, T_g] := T_f T_g - T_g T_f = H^*_{\overline{g}} H_f - H^*_{\overline{f}} H_g.$$

In particular,

$$T_{fg} = T_f T_g, \qquad T_{\overline{g}} T_f = T_{\overline{g} f}$$

when $f \in L^\infty(\Omega)$ and $g \in H^\infty(\Omega)$.

Proof.

$$T_{fg} - T_f T_g = P_+ fg - P_+ f P_+ g = P_+ f(I - P_+)g = H^*_{\overline{f}} H_g;$$

$$[T_f, T_g] = [T_g, T_f) - [T_f, T_g);$$

and $H_g = 0$ if $g \in H^\infty(\Omega)$. $\square$

The following proposition will prove handy when calculating some specific Toeplitz operators on $A^2(\mathbf{D})$.

Proposition 1.3. *Suppose $f \in L^\infty(\mathbf{D})$ is a radial function (i.e. $f(z)$ depends only on $|z|$). Then T_f is a diagonal operator with respect to the basis (1) with weights*

$$c_n(F) := \int_0^1 F(t).(n+1)t^n \, dt,$$

where $F(t) := f(t^{1/2})$, $t \in \langle 0, 1)$.

Proof. Passing to the polar coordinates, we have

$$\langle T_f z^n, z^m \rangle = \int_{\mathbf{D}} f(z)\, z^n \overline{z}^m \, d\nu(z) = \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} f(r) r^{n+m} e^{(n-m)it} \, dt \, 2r \, dr.$$

If $n \neq m$, this is zero; if $n = m$, it equals

$$\int_0^1 F(r^2) r^{2n} \, 2r \, dr = \int_0^1 F(t).t^n \, dt,$$

and the assertion follows immediately. $\square$

The next proposition gives some feeling that the situation for $A^2(\mathbf{D})$ is different from that for H^2 – there are no nonzero compact Toeplitz operators on H^2.

Proposition 1.4. *Assume that $\phi \in L^\infty(\Omega)$ and that the support* $\operatorname{supp} \phi$ *is a compact subset of Ω. Then $M_\phi \restriction A^2(\mathbf{D})$ is a compact operator; consequently, the operators T_ϕ and H_ϕ are also compact.*

Proof. Suppose $\operatorname{supp} \phi = K$ is a compact subset of Ω, and set $R = \operatorname{dist}(K, \mathbf{C}^N \setminus \Omega) > 0$. Assume that $f_n \in A^2(\Omega)$ is a sequence weakly converging to 0. Such sequence must be bounded – say, $\|f\|_2 \le C \quad \forall n$. Then

$$|f_n(\lambda)| \le \|f_n\|_2 \|g_\lambda\|_2 \le \frac{C}{\sqrt{\gamma_N R^{2N}}} \qquad \forall \lambda \in K,$$

whence

$$|\phi(\lambda) f_n(\lambda)| \le \frac{C \|\phi\|_\infty}{\sqrt{\gamma_N R^{2N}}} \qquad \forall \lambda \in \Omega.$$

Also, $f_n \xrightarrow{\mathrm{w}} 0$ implies $f_n(\lambda) = \langle f_n, g_\lambda \rangle \to 0 \quad \forall \lambda \in \Omega$. Thus, we may apply the Lebesgue dominated convergence theorem to conclude that

$$\|\phi f_n\|_2^2 = \int_\Omega |\phi(\lambda) f_n(\lambda)|^2 \, d\lambda = \int_K |\phi(\lambda) f_n(\lambda)|^2 \, d\lambda \to 0$$

as $n \to \infty$. Hence $M_\phi \restriction A^2(\Omega)$ maps weakly convergent sequences into norm convergent ones, and so must be compact. $\square$

Define

$$V(\mathbf{D}) := \{ f \in L^\infty(\mathbf{D}) : \text{ess} \lim_{|z| \nearrow 1} f(z) = 0 \}.$$

Corollary 1.5. *If $\phi \in V(\mathbf{D})$, then $M_\phi \restriction A^2(\mathbf{D})$, T_ϕ and H_ϕ are compact operators.*

Proof. There exist $\phi_n \in V(\mathbf{D})$ such that $\operatorname{supp} \phi_n$ are compact subsets of $\mathbf{D}$ and $\phi_n \rightrightarrows \phi$. Consequently, $M_{\phi_n} \to M_\phi$ in norm; since M_{ϕ_n} are all compact, so must be M_ϕ, and hence also T_ϕ and H_ϕ. $\square$

If we take $\phi \in L^\infty(\Omega)$, $\phi \ge 0$ whose support is a compact subset of Ω, then $\langle T_\phi \mathbf{1}, \mathbf{1} \rangle = \int_\Omega \phi(z) \, dz > 0$ and so $T_\phi \ne 0$; hence, indeed, T_ϕ is nonzero compact Toeplitz operator.

For $\Omega = \mathbf{C}^N$, the space $A^2(\mathbf{C}^N)$, defined as above, would consist only of constant zero; hence, we adopt a different definition in this case. For $x, y \in \mathbf{C}^N$, write

$$\overline{x} y := \sum_{n=1}^N \overline{x_n} y_n \qquad \text{and} \qquad |x| := (\overline{x} x)^{1/2}.$$

(Thus, $|x - y|$ is the usual Euclidean distance between x and y.) The *Gaussian measure* on $\mathbf{C}^N$ is, by definition,

$$d\mu(z) = (2\pi)^{-N} e^{-|z|^2/2} \, dz.$$

Denote $L^p(\mathbf{C}^N, d\mu)$ the usual Lebesgue spaces; $L^\infty(\mathbf{C}^N, d\mu)$ shall be occassionally abbreviated to $L^\infty(\mathbf{C}^N)$, since they happen to coincide. Set, for $1 \le p \le \infty$,

$$A^p(\mathbf{C}^N) := \{ f \in L^p(\mathbf{C}^N, d\mu) : f \text{ is an entire function on } \mathbf{C}^N \}.$$

Again, this is a closed subspace of $L^p(\mathbf{C}^N, d\mu)$. $A^\infty(\mathbf{C}^N) = H^\infty(\mathbf{C}^N)$, which contains only constant zero. For $p = 2$, $A^2(\mathbf{C}^N)$ is a Hilbert space, called the *Fock* or *Segal-Bargmann* space. Many results valid for $A^2(\mathbf{D})$ carry over to the Fock space setting. For a multiindex $n = (n_1, n_2, \ldots, n_N) \in \mathbf{N}^N$, the following abbreviations will be employed:

$$a_n = a_{n_1,n_2,\ldots,n_N},$$

$$z^n = z_1^{n_1} z_2^{n_2} \ldots z_N^{n_N} \qquad (\text{for } z \in \mathbf{C}^N),$$

$$n! = n_1! n_2! \ldots n_N!, \qquad 2^n = 2^{n_1+n_2+\cdots+n_N}.$$

If f is an entire function, $f(z) = \sum_{n\in\mathbf{N}^N} f_n z^n$, then

$$\int_{\mathbf{C}^N} |f(z)|^2 \, d\mu(z) = \sum_{n\in\mathbf{N}^N} n! 2^n |f_n|^2.$$

Consequently, $f \in A^2(\mathbf{C}^N)$ if and only if the last expression is finite. The inner product of f and $g = \sum_{n\in\mathbf{N}^N} g_n z^n$, $f, g \in A^2(\mathbf{C}^N)$, is given by

$$\langle f, g\rangle_{A^2(\mathbf{C}^N)} = \sum_{n\in\mathbf{N}^N} n! 2^n f_n \overline{g_n}.$$

The set

$$\{(n!2^n)^{-1/2} \, z^n\}_{n\in\mathbf{N}^N} \tag{2}$$

is an orthonormal basis of $A^2(\mathbf{C}^N)$. The polynomials are dense in $A^2(\mathbf{C}^N)$. Once again, $A^2(\mathbf{C}^N)$ is a reproducing kernel space; the reproducing kernel at $\lambda \in \mathbf{C}^N$ is given by

$$g_\lambda(z) = e^{\overline{\lambda} z/2},$$

and $\|g_\lambda\|_2 = e^{|\lambda|^2/4}$. For $\phi \in L^\infty(\mathbf{C}^N, d\mu) = L^\infty(\mathbf{C}^N)$, the operators M_ϕ, T_ϕ and H_ϕ may be defined in the same way as for $A^2(\Omega)$; of course, P_+ will be the orthogonal projection of $L^2(\mathbf{C}^N, d\mu)$ onto $A^2(\mathbf{C}^N)$ this time. These operators are bounded, their norms not exceeding $\|\phi\|_\infty$. The maps

$$\phi \mapsto M_\phi, \qquad \phi \mapsto T_\phi, \qquad \phi \mapsto H_\phi$$

are linear. The following propositions are analogies of Propositions 1.2 — 1.5; their proofs are similar, and therefore omitted.

Proposition 1.6. *For arbitrary $f, g \in L^\infty(\mathbf{C}^N)$,*

$$[T_f, T_g) := T_{fg} - T_f T_g = H_{\overline{f}}^* H_g,$$

$$[T_f, T_g] := T_f T_g - T_g T_f = H_{\overline{g}}^* H_f - H_{\overline{f}}^* H_g.$$

In particular,

$$T_{fg} = T_f T_g, \qquad T_{\overline{g}} T_f = T_{\overline{g}f}$$

when $f \in L^\infty(\mathbf{C}^N)$ and $g \in H^\infty(\mathbf{C}^N)$.

Proposition 1.7. *Suppose $f \in L^\infty(\mathbf{C}^N)$ is a radial function (i.e. $f(z)$ depends only on $|z_1|, |z_2|, \dots, |z_n|$). Then T_f is a diagonal operator with respect to the basis (2) with weights*

$$c_n(F) := \int_0^\infty \int_0^\infty \cdots \int_0^\infty F(t) \frac{t^n e^{-(t_1+t_2+\cdots+t_N)}}{n!}\, dt,$$

where $F(t) := f(\sqrt{2t_1}, \sqrt{2t_2}, \dots, \sqrt{2t_N})$, $t \in \langle 0, +\infty)^N$.

Proposition 1.8. *Assume that $\phi \in L^\infty(\mathbf{C}^N)$ and $\operatorname{supp}\phi$ is compact. Then $M_\phi \restriction A^2(\mathbf{C}^N)$, T_ϕ and H_ϕ are compact operators.*

Corollary 1.9. *If $\phi \in V(\mathbf{C}^N)$, where*

$$V(\mathbf{C}^N) := \{f \in L^\infty(\mathbf{C}^N) : \text{ess} \lim_{|z|\to+\infty} f(z) = 0\},$$

then $M_\phi \restriction A^2(\mathbf{C}^N)$, T_ϕ and H_ϕ are compact operators.

The Corollaries 1.5 and 1.9 cannot be inverted — there exist functions $\phi \in L^\infty(\mathbf{D}) \setminus V(\mathbf{D})$ and $\phi \in L^\infty(\mathbf{C}^N) \setminus V(\mathbf{C}^N)$, respectively, such that T_ϕ are compact operators. Furthermore, Toeplitz operators both on $A^2(\mathbf{D})$ and $A^2(\mathbf{C}^N)$ may be well-defined and bounded also for some $\phi \notin L^\infty$. It seems that boundedness or compactness of T_ϕ is not determined by the boundedness of ϕ or its vanishing near the boundary, respectively, but rather by these properties of the image $\widetilde{\phi}$ of ϕ under certain smoothing transformations. For the Fock space, this result is due to Berger and Coburn [6]. If $\phi \in L^2(\mathbf{C}^N, d\mu)$ and $\lambda \in \mathbf{C}^N$, define $k_\lambda := g_\lambda / \|g_\lambda\|$, and

$$\widetilde{\phi}(\lambda) := \langle \phi k_\lambda, k_\lambda \rangle = \int_{\mathbf{C}^N} \phi(z)\, e^{\frac{\overline{\lambda}z}{2} + \frac{\lambda\overline{z}}{2} - \frac{|\lambda|^2}{2}}\, d\mu(z) =$$

$$= \int_{\mathbf{C}^N} \phi(z)\, e^{-|\lambda - z|^2/2} \frac{dz}{(2\pi)^N}.$$

$\widetilde{\phi}$ is called the *Berezin transform* of ϕ; the definition may be extended to $\phi \in L^1(\mathbf{C}^N, d\mu)$ in an obvious way.

Theorem 1.10. *Let $\phi \in L^2(\mathbf{C}^N, d\mu)$.*

(i) If T_ϕ is bounded, $\widetilde{\phi} \in L^\infty(\mathbf{C}^N)$; if it is compact, $\widetilde{\phi} \in V(\mathbf{C}^N)$.

(ii) $M_\phi \restriction A^2(\mathbf{C}^N)$ is bounded iff $(|\phi|^2)^\sim \in L^\infty(\mathbf{C}^N)$, and compact iff $(|\phi|^2)^\sim \in V(\mathbf{C}^N)$.

(iii) If $\phi \geq 0$, then T_ϕ is bounded iff $\widetilde{\phi} \in L^\infty(\mathbf{C}^N)$, and compact iff $\phi \in V(\mathbf{C}^N)$.

For the Bergman space $A^2(\mathbf{D})$, the best reference seems to be Axler [2]. The Möbius transformation ω_λ corresponding to $\lambda \in \mathbf{D}$ is, by definition, the function

$$x \mapsto \omega_\lambda(x) := \frac{x - \lambda}{1 - \overline{\lambda}x},$$

which maps $\mathbf{D}$ bijectively onto itself. The function

$$d(x,y) := |\omega_x(y)|$$

is called the pseudo-hyperbolic metric; it is, indeed, a metric on $\mathbf{D}$. Denote $D_h(\lambda, R)$ the disc with center λ and radius R with respect to this metric. If $\phi \in L^1(\mathbf{D}, d\nu)$, denote

$$\phi^\flat(\lambda) := \frac{1}{\nu(D_h(\lambda, 1/2))} \int_{D_h(\lambda,1/2)} \phi(x)\, d\nu(x),$$

the mean value of ϕ on $D_h(\lambda, 1/2)$.

Theorem 1.11. *Let $\phi \in L^2(\mathbf{D}, d\nu)$.*
(a) $M_\phi \restriction A^2(\mathbf{D})$ is bounded iff $(|\phi|^2)^\flat \in L^\infty(\mathbf{D})$, and compact iff $(|\phi|^2)^\flat \in V(\mathbf{D})$.
(b) If $\phi \geq 0$, T_ϕ is bounded iff $\phi^\flat \in L^\infty(\mathbf{D})$, and compact iff $\phi^\flat \in V(\mathbf{D})$.
The number $1/2$ may be replaced by arbitrary $R \in (0,1)$.

The Berezin transform may be defined on $\mathbf{D}$, too. For $\phi \in L^2(\mathbf{D}, d\nu)$ and $\lambda \in \mathbf{D}$, set $k_\lambda := g_\lambda / \|g_\lambda\|$ and

$$\widetilde{\phi}(\lambda) := \langle \phi k_\lambda, k_\lambda \rangle = \int_{\mathbf{D}} \phi(z) \,.\, \frac{(1-|\lambda|^2)^2}{|1-\overline{\lambda} z|^4}\, d\nu(z).$$

Using the last theorem, it is possible to prove a complete analogue of Theorem 1.10.

Theorem 1.12. *Let $\phi \in L^2(\mathbf{D})$.*
(i) If T_ϕ is bounded, $\widetilde{\phi} \in L^\infty(\mathbf{D})$; if it is compact, $\widetilde{\phi} \in V(\mathbf{D})$.
(ii) $M_\phi \restriction A^2(\mathbf{D})$ is bounded iff $(|\phi|^2)^\sim \in L^\infty(\mathbf{D})$, and compact iff $(|\phi|^2)^\sim \in V(\mathbf{D})$.
(iii) If $\phi \geq 0$, then T_ϕ is bounded iff $\widetilde{\phi} \in L^\infty(\mathbf{D})$, and compact iff $\phi \in V(\mathbf{D})$.

Lemma 1.13. *When $|\lambda| \to 1$, $k_\lambda \xrightarrow{\text{w}} 0$.*

Proof. Since $\|k_\lambda\| = 1 \quad \forall \lambda \in \mathbf{D}$ and the polynomials are dense in $A^2(\mathbf{D})$, it is enough to check that $\langle p, k_\lambda \rangle \to 0$ as $|\lambda| \to 1$ when p is a polynomial; but that's immediate, because

$$\langle z^n, k_\lambda \rangle = (1-|\lambda|^2)\langle z^n, g_\lambda \rangle = (1-|\lambda|^2) \,.\, \lambda^n \to 0 \qquad \text{as } n \to +\infty.$$

$\square$

Remark 1.14. The last lemma remains in force for $A^2(\mathbf{C}^N)$, too; the proof is similar. $\square$

Lemma 1.15. *For each $R \in (0,1)$, there is $C = C(R) > 0$ such that*

$$\int_{\mathbf{D}} \phi(z) \frac{(1-|\lambda|^2)^2}{|1-\overline{\lambda} z|^4}\, d\nu(z) \geq \frac{C^2}{\nu(D_h(\lambda, R))} \int_{D_h(\lambda,R)} \phi(z)\, d\nu(z)$$

for every nonnegative function ϕ on $\mathbf{D}$.

Proof. It suffices to find $C > 0$ so that

$$\frac{(1-|\lambda|^2)^2}{|1-\overline{\lambda}z|^4} \geq \frac{C^2}{\nu(D_h(\lambda,R))} \qquad \forall z \in D_h(\lambda,R).$$

This is equivalent to

$$\frac{1}{|1-\overline{\lambda}z|^2} \geq C.\,\frac{1-R^2|\lambda|^2}{R(1-|\lambda|^2)^2} \qquad \forall z \in D_h(\lambda,R).$$

But, if $z \in D_h(\lambda,R)$,

$$|1-\overline{\lambda}z| = |\lambda|\left|\frac{1}{\overline{\lambda}} - z\right| \leq |\lambda|\left|\frac{1}{\lambda} - \frac{|\lambda|-R}{1-|\lambda|R}\right| = \frac{1-|\lambda|^2}{1-R|\lambda|};$$

hence it is enough to manage that

$$\left(\frac{1-R|\lambda|}{1-|\lambda|^2}\right)^2 \geq C.\frac{1-R^2|\lambda|^2}{R(1-|\lambda|^2)^2},$$

and a short computation reveals that $C = R(1-R)/(1+R)$ will do. □

Proof. (of Theorem 1.12) (i) If T_ϕ is bounded,

$$|\widetilde{\phi}(\lambda)| = |\langle \phi k_\lambda, k_\lambda\rangle| = |\langle T_\phi k_\lambda, k_\lambda\rangle| \leq \|T_\phi\|.\|k_\lambda\|^2 = \|T_\phi\|.$$

If T_ϕ is compact, $\|T_\phi k_\lambda\| \to 0$ as $|\lambda| \to 1$ owing to Lemma 1.13; hence $\widetilde{\phi}(\lambda) = \langle T_\phi k_\lambda, k_\lambda\rangle \to 0$ as $|\lambda| \to 1$.

(ii) $M_\phi \restriction A^2(\mathbf{D})$ is bounded or compact, respectively, if and only if

$$(M_\phi \restriction A^2(\mathbf{D}))^*(M_\phi \restriction A^2(\mathbf{D})) = T_{|\phi|^2}$$

is; hence, (ii) reduces to (iii).

(iii) The "only if" part is contained in (i). To prove the "if" part, suppose that $\phi \geq 0$ and $\widetilde{\phi} \in L^\infty(\mathbf{D})$; then, owing to Lemma 1.15, $\phi^\flat \in L^\infty(\mathbf{D})$, and so T_ϕ is bounded by Theorem 1.11(b). The statement concerning compactness may be proved in a similar manner. □

It is actually possible to go a little further and define the Berezin transform for linear operators on $A^2(\mathbf{C}^N)$ or $A^2(\mathbf{D})$ by the formula

$$\widetilde{T}(\lambda) := \langle Tk_\lambda, k_\lambda\rangle; \qquad k_\lambda = \frac{g_\lambda}{\|g_\lambda\|}, \quad \lambda \in \mathbf{C}^N \text{ or } \mathbf{D}.$$

The operator T need not be bounded, and it suffices if its domain of definition contains all k_λ, $\lambda \in \mathbf{C}^N$ or $\mathbf{D}$.

Proposition 1.16. *(i) If T is bounded, $\widetilde{T} \in L^\infty$ and $\|\widetilde{T}\|_\infty \leq \|T\|$. If T is compact, $\widetilde{T} \in V$.*

(ii) *If* $T = T_\phi$ *is a Toeplitz operator,* $\phi \in L^2$*, then* $\widetilde{T_\phi} = \widetilde{\phi}$*.*

(iii) *If* T *is diagonal with respect to the basis (1) or (2), respectively, then* $\widetilde{T}$ *is radial.*

Here L^∞ is either $L^\infty(\mathbf{D})$ or $L^\infty(\mathbf{C}^N)$, and similarly for L^2 and V.

Proof. (i) $|\widetilde{T}(\lambda)| \leq \|T\|.\|k_\lambda\|^2 = \|T\|$; if T is compact, $Tk_\lambda \to 0$ as $|\lambda| \nearrow 1$ owing to Lemma 1.13, and so $\widetilde{T}(\lambda) \to 0$ as $|\lambda| \to 1-$. The proof for the Fock space is similar (cf. Remark 1.14).

(ii) Immediate.

(iii) We will deal with $A^2(\mathbf{D})$ only, the proof of the other case being similar. Suppose T is diagonal with weights c_n. Then

$$Tg_\lambda = \sum_{n=0}^{\infty} (n+1)\overline{\lambda}^n c_n z^n,$$

whence

$$\widetilde{T}(\lambda) = (1-|\lambda|^2)^2. \sum_{n=0}^{\infty} (n+1) c_n |\lambda|^{2n},$$

and that's a radial function. □

Corollary 1.17. *If* $\phi \in L^2(\mathbf{C}^N)$ *or* $L^2(\mathbf{D})$ *is radial, then so is* $\widetilde{\phi}$*.*

Proof. Combine Propositions 1.3 and 1.7 with Proposition 1.16(iii). □

Remark 1.18. The item (i) of the last proposition bears certain resemblance to the same items of Theorems 1.10 and 1.12. It should be noted, however, that this analogy fails for (iii): there exist unbounded positive operators T such that $\widetilde{T} \in L^\infty$, and non-compact positive operators T such that $\widetilde{T} \in V$. For completeness, we present below an example of the latter (on $A^2(\mathbf{D})$).

Lemma 1.19. *Let* n_k, $k \in \mathbf{N}$, *be an increasing sequence of positive integers such that* $\sum_{k=0}^{\infty} n_k^{-1} < +\infty$. *Let* T *be the operator on* $A^2(\mathbf{D})$ *sending* z^n *into* $c_n z^n$ $(n \in \mathbf{N})$, *where*

$$c_n = \begin{cases} 1 & \text{if } n = n_k - 1 \text{ for some } k \in \mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then T *is a bounded positive noncompact linear operator such that* $\widetilde{T} \in V(\mathbf{D})$.

Proof. Boundedness, positivity and non-compactness are immediate; it remains to show that $\widetilde{T} \in V(\mathbf{D})$. From the proof of the part (iii) of Proposition 1.16, we see that

$$\widetilde{T}(\lambda) = (1-|\lambda|^2)^2. \sum_{k=0}^{\infty} n_k |\lambda|^{2(n_k-1)}.$$

Denote $F(t) = \sum_{k=0}^{\infty} t^{n_k}$. Then $F \in L^1(0,1)$, since

$$\int_0^1 F(t)\, dt = \sum_{k=0}^{\infty} \frac{t^{n_k+1}}{n_k+1}\bigg|_{t=0}^{1} = \sum_{k=0}^{\infty} \frac{1}{n_k+1},$$

which is finite by assumption. Consequently, the integral

$$\int_t^1 F(s)\,ds$$

must tend to zero as $t \nearrow 1$. Because F is a nondecreasing function on $(0,1)$,

$$\int_t^1 F(s)\,ds \geq (1-t)F(t),$$

hence also $\lim_{t\to 1-}(1-t)F(t) = 0$. On the other hand, it follows from the l'Hopital rule that

$$\lim_{t\to 1-} \frac{F(t)}{(1-t)^{-1}} = -\lim_{t\to 1-} \frac{F'(t)}{(1-t)^{-2}} = -\lim_{t\to 1-} (1-t)^2.\sum_{k=0}^{\infty} n_k t^{n_k - 1}.$$

Substituting $|\lambda|^2$ for t leads to the desired conclusion. $\square$

If H is a Hilbert space, $\mathcal{B}(H)$ stands for the C^*-algebra of all bounded linear operators on H (with the operator norm $\|\cdot\|$), $\mathtt{Comp}(H)$ for the space of all compact operators from $\mathcal{B}(H)$; $\mathtt{Comp}(H)$ will be abbreviated $\mathtt{Comp}$ when it's clear what H is. The norm is denoted $\|\cdot\|$ on all spaces; $\|\cdot\|_p$ is sometimes used for the norm in L^p or A^p. Where ambiguity might arise, the space where the norm is taken is given as an index, e.g. $\|f\|_{L^2(\mathbf{D},d\nu)}$. Similar conventions will be observed for scalar products $\langle\,.\,,\,.\,\rangle$. SOT and WOT are abbreviations for the strong and weak operator topologies on $\mathcal{B}(H)$, respectively. If $\mathcal{M}$ is a subset of $\mathcal{B}(H)$, $\operatorname{clos}\mathcal{M}$ denotes the closure of $\mathcal{M}$ in the operator norm topology. When discussing $\mathcal{B}(A^2(\mathbf{D}))$ or $\mathcal{B}(A^2(\mathbf{C}^N))$, $\operatorname{diag}(c_n)$ means the diagonal operator with weights c_n with respect to the basis (1) or (2), respectively. The symbol $\rightrightarrows$ is employed to denote uniform convergence; $\xrightarrow{\text{w}}$ means weak convergence in a Banach space. A bar over a subset of $\mathbf{C}^N$ denotes the closure; a bar over a number, a function, etc., the complex conjugate. For other symbols, not mentioned in this Introduction, consult the List of Notation.

By a *Bergman-type space,* we mean either a Bergman space $A^2(\Omega)$, $\Omega \subset \mathbf{C}^N$, or the Fock space $A^2(\mathbf{C}^N)$.

Chapter 2. DENSITY OF TOEPLITZ OPERATORS

Let T_ϕ be a Toeplitz operator on the Hardy space H^2. It is easily seen that

$$T_z^* T_\phi T_z = T_\phi \qquad \text{for any } \phi \in L^\infty(\mathbf{T}).$$

According to a classical result, the converse also holds: if some operator $T : H^2 \longrightarrow H^2$ satisfies $T_z^* T T_z = T$, then $T = T_\phi$ for some $\phi \in L^\infty(\mathbf{T})$. This result serves as a starting point for the theory of symbols of operators (cf. [24]). It shows that, loosely speaking, there are only few Toeplitz operators on H^2.

In my paper [15], I have shown that Toeplitz operators on $A^2(\mathbf{D})$ don't admit the characterization as above. More precisely, if $AT_\phi B = T_\phi$ for all $\phi \in L^\infty(\mathbf{D})$, then $A = cI$, $B = c^{-1}I$ for some (nonzero) complex number c. The proof works also for $A^2(\Omega)$ and $A^2(\mathbf{C}^N)$.

A natural question to ask is if this is not because there are, loosely speaking, more operators which are Toeplitz than in the classical (i.e. H^2) case. To put it precisely, we can ask if following statements are true:

(1°) the Toeplitz operators are dense (in some topology) in $\mathcal{B}(A^2(\mathbf{D}))$;

(2°) every finite dimensional operator is Toeplitz;

(3°) for any linearly independent elements $f, g \in A^2(\mathbf{D})$, there exists $\phi \in L^\infty(\mathbf{D})$ such that $T_\phi f = g$.

Clearly (2°) implies (1°) in the strong operator topology, and either (1°) or (3°) implies the impossibility of the above-mentioned characterization $AT_\phi B = T_\phi$.

The statement (2°) is easily seen not to be true. For example, there is no $\phi \in L^\infty(\mathbf{D})$ such that $T_\phi = \langle\,.\,, \mathbf{1}\rangle\mathbf{1}$. This fact is an easy consequence of the Müntz-Szász theorem for L^2 spaces (see, for instance, [14]). In fact, this theorem yields a much stronger result: if $T_\phi = \langle\,.\,, f\rangle g$ for some $\phi \in L^\infty(\mathbf{D})$ and $f, g \in A^2(\mathbf{D})$, $f(z) = \sum_0^\infty f_n z^n$, $g(z) = \sum_0^\infty g_n z^n$, then

$$\sum_{f_n=0} \frac{1}{n} < +\infty \qquad \text{and} \qquad \sum_{g_n=0} \frac{1}{n} < +\infty\,.$$

(Loosely speaking, only few Taylor coefficients of f and g can be zero.) It's a conjecture of author's that in fact there are no finite-dimensional Toeplitz operators in $\mathcal{B}(A^2(\mathbf{D}))$ at all.

(3°) is readily seen to be false, too. It suffices to take $f = \mathbf{1}$: if there were, for every $g \in A^2(\mathbf{D})$, some $\phi \in L^\infty(\mathbf{D})$ such that $g = T_\phi \mathbf{1}\ (= P_+\phi)$, then the mapping (here $(A^2(\mathbf{D}))^d$ stands for the dual space of $A^2(\mathbf{D})$)

$$A : L^\infty(\mathbf{D}) \longrightarrow (A^2(\mathbf{D}))^d\ , \quad \phi \longmapsto \langle\,.\,, P_+\overline{\phi}\rangle = \langle\,.\,, \overline{\phi}\rangle_{L^2(\mathbf{D})}$$

would be onto. Let B be the operator of inclusion of $A^2(\mathbf{D})$ into $L^1(\mathbf{D})$:

$$B : A^2(\mathbf{D}) \longrightarrow L^1(\mathbf{D})\ , \quad \psi \longmapsto \psi\,.$$

This is a continuous operator (by the Schwarz inequality) and has A as its adjoint, $B^d = A$. By the Hausdorff normal solvability theorem (cf. [26], chapter VII, §5),

$\operatorname{Ran} A$ is closed if, and only if, $\operatorname{Ran} B$ is closed. Because B is injective, $\operatorname{Ran} B$ is closed if and only if B is bounded below (just use the open mapping theorem). But the norm of z^n in $L^1(\mathbf{D})$ is

$$\int_{\mathbf{D}} |z^n|\, dz = \int_0^1 r^n.2r\, dr = \frac{2}{n+2}\,,$$

whereas the norm of z^n in $A^2(\mathbf{D})$ is

$$\|z^n\|_2 = \left(\int_{\mathbf{D}} |z^n|^2\, dz\right)^{1/2} = (n+1)^{-1/2}\,.$$

Consequently, B is not bounded below, so $\operatorname{Ran} A$ is not closed, and A cannot be surjective. (The last part of the argument can be avoided by evoking directly the fact that the closure of $A^2(\mathbf{D})$ in $L^1(\mathbf{D})$ is $A^1(\mathbf{D})$, the space of all integrable analytic functions on $\mathbf{D}$. Our method is more elementary.)

All the same, there is a weakened version of (3°) that does hold and which, moreover, implies that (1°) holds in SOT.

Theorem 2.1. *Let*[3] $\Omega \subseteq \mathbf{C}^N$, $T \in \mathcal{B}(A^2(\Omega))$, $F_i, G_i \in A^2(\Omega)$ $(i = 1, 2, \ldots, N)$. *Then there exists* $\phi \in L^\infty(\Omega)$ *such that*

$$\langle T_\phi F_i, G_i\rangle = \langle TF_i, G_i\rangle \quad , \quad i = 1, 2, \ldots, N.$$

Proof. We shall prove the theorem for the case $\Omega \neq \mathbf{C}^N$; the proof for the Fock space is perfectly similar. Let $f_1, f_2, \ldots, f_n$, resp. $g_1, g_2, \ldots, g_m$ be a basis of the (finite-dimensional) subspace of $A^2(\Omega)$ generated by $F_1, \ldots, F_N$, resp. $G_1, \ldots, G_N$. Clearly it's sufficient to find $\phi \in L^\infty(\Omega)$ such that

$$\langle T_\phi f_i, g_j\rangle = \langle Tf_i, g_j\rangle \qquad \text{for all} \quad i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

Consider the operator $R : L^\infty(\Omega) \longrightarrow \mathbf{C}^{n\times m}$, defined by the formula

$$(R\phi)_{ij} = \int_\Omega \phi(z) f_i(z)\overline{g_j(z)}\, dz = \langle T_\phi f_i, g_j\rangle.$$

Suppose some $u \in \mathbf{C}^{n\times m}$ is orthogonal to the range of R, i.e.

$$\sum_{i=1}^n \sum_{j=1}^m (R\phi)_{ij}\overline{u}_{ij} = 0 \qquad \text{for all} \quad \phi \in L^\infty(\Omega).$$

This means that

$$\int_\Omega \phi(z). \sum_{i=1}^n \sum_{j=1}^m \overline{u}_{ij} f_i(z)\overline{g_j(z)}\, dz = 0 \qquad \text{for all} \quad \phi \in L^\infty(\Omega), \tag{3}$$

[3]Remember that, according to the convention from the Introduction, this means that either Ω is a domain in $\mathbf{C}^N$, or that $\Omega = \mathbf{C}^N$.

which implies

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\overline{u}_{ij}f_i(z)\overline{g_j(z)} = 0 \tag{4}$$

dz-almost everywhere in Ω; since the left-hand side is obviously continuous on Ω, this equality holds, in fact, on the whole of Ω. Thus, the function

$$F(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{u}_{ij}f_i(x)\overline{g_j(\overline{y})},$$

which is analytic in $\Omega\times\Omega$, equals zero whenever $x=\overline{y}$. By a well-known uniqueness theorem, this implies that F is identically zero on $\Omega\times\Omega$. Because the functions f_i, $i=1,2,\dots,n$, are linearly independent, we have

$$\sum_{j=1}^{m} u_{ij}g_j(\overline{y}) = 0 \qquad \text{for all } y\in\Omega,\ i=1,2,\dots,n;$$

but g_j, $j=1,2,\dots,m$, are also linearly independent, and so $u_{ij}=0$ for all i,j, i.e. $u=0$. This means that the range of R is all of $\mathbf{C}^{n\times m}$, which immediately yields the desired conclusion. $\square$

Corollary 2.2. *The set $\mathcal{T}=\{T_\phi : \phi\in L^\infty(\Omega)\}$ is dense in $\mathcal{B}(A^2(\Omega))$ in SOT (the strong operator topology). The assertion holds also for $\Omega=\mathbf{C}^N$.*

Proof. In view of the preceding theorem, it is certainly dense in WOT (the weak operator topology); and because $\mathcal{T}$ is a subspace, i.e. a convex set, its WOT- and SOT- closures coincide. $\square$

Note that the crucial step in the proof of Theorem 2.1 was the implication (3)⇒(4). Thus, the theorem remains in force if we have (3) only for $\phi\in\mathcal{D}(\Omega)$ (the set of all infinitely differentiable functions on Ω whose support is a compact subset of Ω) — any w^*-dense subset of $L^\infty(\Omega)$ will do. Consequently, the following theorem holds.

Theorem 2.3. *Let $\Omega\subseteq\mathbf{C}^N$. Then the set $\mathcal{T}_1=\{T_\phi : \phi\in\mathcal{D}(\Omega)\}$ is SOT-dense in $\mathcal{B}(A^2(\Omega))$.*

A natural question arises at this point — namely, whether the Toeplitz operators are not actually *norm-dense* in $\mathcal{B}(A^2(\Omega))$. We shall see later that this is not the case — even the C^*-algebra generated by them is smaller than $\mathcal{B}(A^2(\Omega))$. It is true, however, that the norm closure of the Toeplitz operators contains all compact operators.

Theorem 2.4. *Suppose $\Omega\subseteq\mathbf{C}^N$. Let $\mathcal{T}_1=\{T_\phi : \phi\in\mathcal{D}(\Omega)\}$. Then* $\operatorname{clos}\mathcal{T}_1 = \mathtt{Comp}(A^2(\Omega))$.

Since $T_\phi\in\mathtt{Comp}$ for $\phi\in\mathcal{D}(\Omega)$ in view of Propositions 1.4 and 1.8, the inclusion $\operatorname{clos}\mathcal{T}_1\subset\mathtt{Comp}(A^2(\Omega))$ is obvious; it remains to show that

$$\mathtt{Comp}(A^2(\Omega))\subset\operatorname{clos}\mathcal{T}_1.$$

We are going to present two proofs. The first is shorter and works for arbitrary $\Omega \subseteq \mathbf{C}^N$; however, it is not constructive. The second applies only for $\mathbf{D}$ or $\mathbf{C}^N$, involves much computation, but exhibits explicitely how to manage the approximation.

Proof. FIRST (SHORT) PROOF. We begin by a simple lemma.

Lemma 2.5. *Let $\Omega \subseteq \mathbf{C}^N$. Then the reproducing kernels g_λ, $\lambda \in \Omega$, span $A^2(\Omega)$.*

Proof. Suppose $f \in A^2(\Omega)$ is orthogonal to all g_λ, $\lambda \in \Omega$. Then $f(\lambda) = \langle f, g_\lambda \rangle = 0\ \forall \lambda \in \Omega$, i.e. $f = 0$. $\square$

Recall that the dual of $\mathtt{Comp}(H)$, where H is a separable Hilbert space, may be identified with $\mathtt{Trace}(H)$, the space of all trace class operators on H equipped with the trace norm $\|\cdot\|_{\mathrm{Tr}}$; the pairing is given by $(K,T) \mapsto \mathrm{Tr}(KT) = \mathrm{Tr}(TK)$, Tr being the trace.

Suppose that $\Omega \subseteq \mathbf{C}^N$ and that $\operatorname{clos} \mathcal{T}_1$ is a proper subset of $\mathtt{Comp}(A^2(\Omega))$. By the Hahn-Banach theorem, there exists $T \in \mathtt{Trace}(A^2(\Omega))$, $T \neq 0$, such that

$$\mathrm{Tr}(TT_\phi) = 0 \qquad \forall \phi \in \mathcal{D}(\Omega).$$

Let A, B be two Hilbert-Schmidt operators such that $T = AB^*$, $\{e_n\}_{n=0}^\infty$ an orthonormal basis for $A^2(\Omega)$, $f_n = Ae_n$, $g_n = Be_n$. Then $\mathrm{Tr}(TT_\phi) = \mathrm{Tr}(B^* T_\phi A) = \sum\limits_{n=0}^{\infty} \langle B^* T_\phi A e_n, e_n \rangle$, and so the last condition may be rewritten as

$$\sum_{n=0}^{\infty} \langle T_\phi f_n, g_n \rangle = 0 \qquad \forall \phi \in \mathcal{D}(\Omega),$$

or

$$\sum_{n=0}^{\infty} \int_\Omega \phi(\lambda) f_n(\lambda) \overline{g_n(\lambda)}\, d\nu(\lambda) = 0 \qquad \forall \phi \in \mathcal{D}(\Omega)$$

(if $\Omega = \mathbf{C}^N$, replace $d\nu$ by $d\mu$). Because ϕ has compact support, $\sum \|f_n\|^2 < +\infty$ and $\sum \|g_n\|^2 < +\infty$, we may interchange the integration and summation signs, which yields

$$\int_\Omega \phi(\lambda) F(\lambda, \overline{\lambda})\, d\nu(\lambda) \text{ (or } d\mu(\lambda)) = 0 \qquad \forall \phi \in \mathcal{D}(\Omega),$$

where

$$F(x,y) = \sum_{n=0}^{\infty} f_n(x) \overline{g_n(\overline{y})} = \mathrm{Tr}(TG_{x,y}),$$

$$G_{x,y} = \langle \cdot\, , g_x \rangle g_{\overline{y}}.$$

It follows that $F(\lambda, \overline{\lambda}) = 0$ for almost all $\lambda \in \Omega$; in other words, the function $F(x,y)$, analytic on $\Omega \times \Omega$, vanishes when $x = \overline{y}$. Appealing to the aforementioned uniqueness principle, $F = 0$ everywhere on $\Omega \times \Omega$, i.e. $\mathrm{Tr}(TG_{x,y}) = 0 \quad \forall x, y \in \Omega$. In view of the last lemma, $\mathrm{Tr}(TK) = 0$ for all rank one operators K; by linearity and continuity, $\mathrm{Tr}(TK) = 0$ for all compact K, whence $T = 0$ — a contradiction. The proof is complete. $\square$

The other proof is a slightly simplified version of the author's original one from [16]. It is based on two lemmas. We state and prove them first for $A^2(\mathbf{D})$; the (slightly easier) case of $A^2(\mathbf{C}^N)$ will be discussed afterwards.

For $m, n \in \mathbf{N}$ and $a \in \mathbf{D}$, let $T_{(m,n,a)}$ be the operator on $A^2(\mathbf{D})$ given by

$$f \longmapsto \langle f, g_{m,a} \rangle g_{n,a}\,,$$

where $a \in \mathbf{D}$, m and n are non-negative integers, and $g_{m,a} \in A^2(\mathbf{D})$ is given by the formula

$$g_{m,a}(x) := \frac{(m+1)!z^m}{(1-\overline{a}z)^{m+2}}.$$

One has

$$\langle f, g_{m,a} \rangle = f^{(m)}(a),$$

the m-th derivative of $f \in A^2(\mathbf{D})$ at $a \in \mathbf{D}$; thus,

$$\langle T_{(m,n,a)} f, g \rangle = f^{(m)}(a)\overline{g^{(n)}(a)}$$

for arbitrary $f, g \in A^2(\mathbf{D})$.

Lemma 2.6. *Let M, N be non-negative integers, $a \in \mathbf{D}$, and denote*

$$R_{(M,N,a,t)} = \frac{T_{(M,N,a+t)} - T_{(M,N,a)}}{2t} - i\frac{T_{(M,N,a+it)} - T_{(M,N,a)}}{2t}\ .$$

Then $R_{(M,N,a,t)}$ tends to $T_{(M+1,N,a)}$ (in norm) as the real number t tends to zero:

$$\lim_{\mathbf{R}\ni t\to 0} \left\| R_{(M,N,a,t)} - T_{(M+1,N,a)} \right\| = 0\,.$$

Similarly,

$$\lim_{\mathbf{R}\ni t\to 0} \left\| R'_{(M,N,a,t)} - T_{(M,N+1,a)} \right\| = 0\,,$$

where

$$R'_{(M,N,a,t)} = \frac{T_{(M,N,a+t)} - T_{(M,N,a)}}{2t} + i\frac{T_{(M,N,a+it)} - T_{(M,N,a)}}{2t}\ .$$

(Here i stands for $\sqrt{-1}$.)

Proof. Let F, $G \in A^2(\mathbf{D})$ and denote, for a while, $f = F^{(M)}$ and $g = G^{(N)}$. Then

$$\left\langle \left(R_{(M,N,a,t)} - T_{(M+1,N,a)}\right) F, G \right\rangle =$$

(5)

$$= \frac{f(a+t)\overline{g(a+t)} - f(a)\overline{g(a)}}{2t} - i\frac{f(a+it)\overline{g(a+it)} - f(a)\overline{g(a)}}{2t} - f'(a)\overline{g(a)}\,.$$

Let

$$f(x) = \sum_0^\infty f_n.(x-a)^n\,, \quad g(x) = \sum_0^\infty g_n.(x-a)^n \tag{6}$$

be the Taylor expansion of f, resp. g at a. These series are locally uniformly convergent in the disc $|x-a|<1-a$. Consequently, for $|t|<1-|a|$ the right-hand side of (5) equals to

$$\frac{1}{2t}\left(\sum_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} f_m t^m.\overline{g}_n t^n\right)-\frac{i}{2t}\left(\sum_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} f_m.(it)^m.\overline{g}_n.(-it)^n\right)-f_1\overline{g}_0\,.$$

Rearranging all terms into one series, the terms corresponding to $(m,n)=(1,0)$ and $(0,1)$ cancel, and we get

$$\frac{1}{2}\cdot\sum_{\substack{m,n\geq 0\\m+n\geq 2}} f_m\overline{g}_n t^{m+n-1}(1-i^{m-n+1})\,. \tag{7}$$

Let F_n, resp. G_n be the coefficients of the Taylor expansion of the function F, resp. G at a. Because $f=F^{(M)}$, we have

$$f_m=\frac{1}{m!}f^{(m)}(a)=\frac{1}{m!}F^{(M+m)}(a)=\frac{(M+m)!}{m!}F_{m+M}\,,$$

and similarly for g_n. It follows that (7) is equal to

$$\frac{1}{2}\cdot\sum_{\substack{m,n\geq 0\\m+n\geq 2}}\frac{(M+m)!}{m!}F_{m+M}\cdot\frac{(N+n)!}{n!}\overline{G}_{n+N}.t^{m+n-1}.(1-i^{m-n+1})\,. \tag{8}$$

We are going to estimate the absolute value of the last expression in terms of $\|F\|_2$ and $\|G\|_2$. One has

$$\begin{aligned}\|F\|_2^2=\int_{\mathbf{D}}|F(z)|^2\,dz &\geq \int\limits_{|z-a|<1-|a|}\|F(z)\|^2\,dz=\\ &=\int_0^{1-|a|}\int_0^{2\pi}\sum_{j,k\geq 0}F_j\overline{F}_k r^{j+k}e^{(j-k)it}.\frac{r}{\pi}\,dt\,dr\ ,\end{aligned}$$

(we have passed to polar coordinates). Since the Taylor series

$$F(z)=\sum_0^\infty F_n.(z-a)^n$$

is locally uniformly convergent on the disc $|z-a|<1-|a|$, we can interchange the integration and summation signs and get

$$\|F\|_2^2\geq\sum_{j,k=0}^\infty\int_0^{1-|a|}\int_0^{2\pi}F_j\overline{F}_k r^{j+k}e^{(j-k)it}.\frac{r}{\pi}\,dr\,dt=$$

$$=\sum_{k=0}^\infty\frac{(1-|a|)^{2k+2}}{k+1}\,|F_k|^2\,. \tag{9}$$

Similar argument holds for G. Denote, for a while,

$$\alpha_k = \frac{(1-|a|)^{k+1}}{(k+1)^{1/2}}\,|F_k|\,,\quad \beta_k = \frac{(1-|a|)^{k+1}}{(k+1)^{1/2}}\,|G_k|\,.$$

According to (9), α and β belong to l^2 and

$$\|\alpha\|_2 \le \|F\|_2\,,\quad \|\beta\|_2 \le \|G\|_2\,.$$

Returning to our previous calculations, we see that the absolute value of (8) is not greater then

$$\frac{1}{2}\cdot\sum_{\substack{m,n\ge 0\\ m+n\ge 2}} \frac{(M+m)!}{m!}\,|F_{m+M}|\,.\,\frac{(N+n)!}{n!}\,|G_{n+N}|\,.\,t^{m+n-1}\,.\,2 =$$

$$= \sum_{\substack{m,n\ge 0\\ m+n\ge 2}} \frac{(M+m)!}{m!}\,\frac{(M+m+1)^{1/2}t^{m-\frac{1}{2}}}{(1-|a|)^{M+m+1}}\,\alpha_{M+m}\,.\,\frac{(N+n)!}{n!}\,\frac{(N+n+1)^{1/2}t^{n-\frac{1}{2}}}{(1-|a|)^{N+n+1}}\,\beta_{N+n}\,.$$

Break this sum into three parts, namely,

$$\sum_{m,n\ge 1} + \sum_{\substack{m=0\\ n\ge 2}} + \sum_{\substack{m\ge 2\\ n=0}} := \sigma_1+\sigma_2+\sigma_3\,.$$

Let's first consider σ_1. Obviously

$$\begin{aligned} \sigma_1 &= \left(\sum_{m=1}^{\infty}\frac{(M+m)!}{m!}\,\frac{(M+m+1)^{\frac{1}{2}}t^{m-\frac{1}{2}}}{(1-|a|)^{M+m+1}}\,\alpha_{M+m}\right)\times \\ &\times\left(\sum_{n=1}^{\infty}\frac{(N+n)!}{n!}\,\frac{(N+n+1)^{\frac{1}{2}}t^{n-\frac{1}{2}}}{(1-|a|)^{N+n+1}}\,\beta_{N+n}\right)\,. \end{aligned} \tag{10}$$

According to the Cauchy-Schwarz inequality, the first factor on the right-hand side is less than or equal to

$$\begin{aligned} &\|\alpha\|_2\,.\left(\sum_{m=1}^{\infty}\frac{(M+m)!^2}{m!^2}\,.\,\frac{(M+m+1).t^{2m-1}}{(1-|a|)^{2M+2m+2}}\right)^{1/2} = \\ &= \|\alpha\|_2\,.\left(t\,.\sum_{m=0}^{\infty}\frac{(M+m+1)!^2}{(m+1)!^2}\,.\,\frac{(M+m+2).t^{2m}}{(1-|a|)^{2M+2m+4}}\right)^{1/2} = \\ &\|\alpha\|_2\,.\,t^{1/2}c_1(M,a,t), \end{aligned} \tag{11}$$

where $c_1(M,a,t)$ tends to a finite limit $\dfrac{(M+1)!^2(M+2)}{(1-|a|)^{2M+4}}$ as $t^2\to 0+$, i.e. as $t\to 0$.

A similar estimate holds for the second factor in (10). Putting the two estimates together, we see that

$$\begin{aligned}\sigma_1 &\le \|\alpha\|_2 \,.\, \|\beta\|_2 \,.\, t \,.\, c_2(M,N,a,t) \\ &\le \|F\|_2 \,.\, \|G\|_2 \,.\, t \,.\, c_2(M,N,a,t)\,,\end{aligned} \tag{12}$$

where

$$c_2(M,N,a,t) = c_1(M,a,t)c_1(N,a,t)$$

tends to a finite limit as $t \to 0$.

Now let's turn our attention to σ_2. We have

$$\sigma_2 \le \frac{M!(M+1)^{1/2}}{(1-|a|)^{M+1}} \,.\, \|\alpha\|_2 \,.\, \left[\sum_{n=2}^{\infty} \frac{(N+n)!}{n!} \, \frac{(N+n+1)^{1/2}}{(1-|a|)^{N+n+1}} \,.\, t^{n-1} \,.\, \beta_{N+n}\right].$$

Using Cauchy-Schwarz inequality shows that the bracketed term is not greater then

$$\|\beta\|_2 \,.\, \left(\sum_{n=2}^{\infty} \frac{(N+n)!^2}{n!^2} \,.\, \frac{(N+n+1)}{(1-|a|)^{2N+2n+2}} \,.\, t^{2n-2}\right)^{1/2} = \|\beta\|_2 \,.\, tc_3(N,a,t)\,, \tag{13}$$

where $c_3(N,a,t)$ tends to a finite limit as $t \to 0$. Consequently,

$$\begin{aligned}\sigma_2 &\le c_4(M,N,a,t) \,.\, \|\alpha\|_2 \,.\, \|\beta\|_2 \,.\, t \le \\ &\le c_4(M,N,a,t) \,.\, \|F\|_2 \,.\, \|G\|_2 \,.\, t\,,\end{aligned}$$

with $c_4(M,N,a,t)$ tending to a finite limit as $t \to 0$.

Similar estimate, of course, can be obtained for σ_3. Summing up,we see that

$$|\langle (R_{(M,N,a,t)} - T_{(M+1,N,a)})F, G\rangle| \le c_5(M,N,a,t) \,.\, t \,.\, \|F\|_2 \,.\, \|G\|_2$$

for all F, $G \in A^2(\mathbf{D})$, where $c_5(M,N,a,t)$ tends to a finite limit as $t \to 0$. Consequently,

$$\|R_{(M,N,a,t)} - T_{(M+1,N,a)}\| \le c_5(M,N,a,t) \,.\, t$$

and the first part of the lemma follows. The assertion concerning $R'_{(M,N,a,t)}$ can be proved in the same way. $\square$

We need one more lemma, the proof of which is (fortunately) a little shorter. Remember the symbol "clos" denotes the closure in the norm topology of $\mathcal{B}(A^2(\mathbf{D}))$.

Lemma 2.7. *Denote $\mathcal{T}_1 = \{T_\phi : \phi \in \mathcal{D}(\mathbf{D})\}$. Then*

$$T_{(0,0,a)} \in \operatorname{clos} \mathcal{T}_1 \qquad \text{for every} \;\; a \in \mathbf{D}\,.$$

Proof. For each δ in the interval $(0, \frac{1-|a|}{2})$, pick a function $f_\delta \in \mathcal{D}(\mathbf{D})$ such that

$$f_\delta(z) = \begin{cases} 0 & \text{if} \quad |z-a| \ge \delta + \delta^2 \\ \delta^{-2} & \text{if} \quad |z-a| \le \delta \end{cases}$$

and

$$0 \le f_\delta(z) \le \delta^{-2} \qquad \text{if} \quad \delta < |z-a| < \delta + \delta^2 \,.$$

Let $f,\, g \in a^2$. Then

$$\langle (T_{f_\delta} - T_{(0,0,a)})f, g\rangle = \int\limits_{\mathbf{D}} f_\delta(z) f(z)\overline{g(z)}\, dz - f(a)\overline{g(a)} =$$

$$= \left[\frac{1}{\delta^2} \int\limits_{|z-a|\le\delta} f(z)\overline{g(z)}\, dz - f(a)\overline{g(a)} \right] + \int\limits_{\delta<|z-a|<\delta+\delta^2} f_\delta(z) f(z)\overline{g(z)}\, dz :=$$

(14)

$$:= \rho_1 + \rho_2 \,.$$

Let f_n, resp. g_n be the coefficients of the Taylor expansion of f, resp. g at a:

$$f(x) = \sum_0^\infty f_n \,.\, (x-a)^n \,, \quad g(x) = \sum_0^\infty g_n \,.\, (x-a)^n \,.$$

Substituting these formulas into the expression for ρ_1 gives

$$\rho_1 = \frac{1}{\delta^2} \int\limits_{|z-a|\le\delta} \sum_{m,n=0}^\infty f_n \,.\, (z-a)^n \,.\, \overline{g}_m \,.\, \overline{(z-a)^m}\; dz \; - f_0\overline{g}_0 =$$

$$= \frac{1}{\delta^2} \int_0^\delta \int_0^{2\pi} \sum_{m,n=0}^\infty f_n \overline{g}_m r^{n+m} e^{(n-m)it} \,.\, \frac{r}{\pi}\, dt\, dr \; - f_0\overline{g}_0 =$$

$$= \frac{1}{\delta^2} \,.\, \sum_{n=0}^\infty f_n \overline{g}_n \,.\, \frac{\delta^{2n+2}}{n+1} \; - f_0\overline{g}_0 =$$

$$= \sum_{n=1}^\infty f_n \overline{g}_n \,.\, \frac{\delta^{2n}}{n+1} \,.$$

Denote, for a little while,

$$\alpha_n = \frac{(1-|a|)^{n+1}}{(n+1)^{1/2}} \, |f_n| \,, \quad \beta_n = \frac{(1-|a|)^{n+1}}{(n+1)^{1/2}} \, |g_n| \,.$$

In course of the proof of Lemma 2.6, we have seen that α and β belong to l^2 and

$$\|\alpha\|_2 \le \|f\|_2 \,, \quad \|\beta\|_2 \le \|g\|_2 \,.$$

Now

$$\sum_{n=1}^\infty |f_n \overline{g}_n| \,.\, \frac{\delta^{2n}}{n+1} = \sum_{n=1}^\infty \alpha_n \beta_n \,.\, \frac{\delta^{2n}}{(1-|a|)^{2n+2}} =$$

$$= \frac{\delta^2}{(1-|a|)^4} \,.\, \sum_{n=1}^\infty \alpha_n \beta_n \,.\, \left[\frac{\delta}{1-|a|} \right]^{2(n-1)} \le \frac{\delta^2}{(1-|a|)^4} \,.\, \sum_{n=1}^\infty \alpha_n \beta_n \le$$

$$\le \frac{\delta^2}{(1-|a|)^4} \,.\, \|\alpha\|_2 \,.\, \|\beta\|_2 \le \frac{\delta^2}{(1-|a|)^4} \,.\, \|f\|_2 \,.\, \|g\|_2 \,,$$

which implies that

$$|\rho_1| \le \frac{\delta^2}{(1-|a|)^4} \,.\, \|f\|_2 \,.\, \|g\|_2 \,. \tag{15}$$

As for ρ_2, we have

$$|\rho_2| \le \sup_{\delta<|z-a|<\delta+\delta^2} |f_\delta(z) f(z) \overline{g(z)}| \,.\, \int\limits_{\delta<|z-a|<\delta+\delta^2} dz \le$$
$$\le \frac{1}{\delta^2} \cdot \sup_{\delta<|z-a|<\delta+\delta^2} |f(z)g(z)| \,.\, \delta^3(\delta+2) \;.$$

Because

$$|f(z)| = |\langle f, g_z\rangle| \le \|f\|_2 . \, \|g_z\|_2 = \|f\|_2 . \, [g_z(z)]^{1/2} = \frac{\|f\|_2}{1-|z|^2}$$

(and similarly for g), the supremum does not exceed

$$\frac{\|f\|_2 . \, \|g\|_2}{[1-(|a|+\delta+\delta^2)^2]^2} \;.$$

Summing up, we have

$$|\rho_2| \le \delta \,.\, \|f\|_2 . \, \|g\|_2 . \, c_6(a,\delta), \tag{16}$$

where

$$c_6(a,\delta) = \frac{\delta+2}{[1-(|a|+\delta+\delta^2)^2]^2}$$

tends to a finite limit as $\delta \to 0+$.

Putting together (14),(15) and (16) yields

$$\| T_{f_\delta} - T_{(0,0,a)} \| \le c_7(a,\delta) \,.\, \delta \,,$$

where $c_7(a,\delta)$ tends to a finite limit as $\delta \to 0+$. The lemma follows immediately. $\square$

Proof. Second (constructive) proof (of Theorem 2.4). Note that the mapping $\phi \mapsto T_\phi$ is linear, so $\mathcal{T}_1$ and $\operatorname{clos}\mathcal{T}_1$ are linear subsets (i.e. subspaces) of $\mathcal{B}(A^2(\mathbf{D}))$. In view of Lemma 2.7, $T_{(0,0,a)} \in \operatorname{clos}\mathcal{T}_1$ for each $a \in \mathbf{D}$. By linearity, $R_{(0,0,a,t)}$ and $R'_{(0,0,a,t)} \in \operatorname{clos}\mathcal{T}_1$ whenever $a \in \mathbf{D}$ and $|t| < 1-|a|$; by Lemma 2.6, this implies $T_{(1,0,a)}$ and $T_{(0,1,a)} \in \operatorname{clos}\mathcal{T}_1$. Proceeding by induction, we conclude that $T_{(m,n,a)} \in \operatorname{clos}\mathcal{T}_1$ for every $a \in \mathbf{D}$ and $m,n = 0,1,2,\ldots$. Taking $a = 0$ shows that, in particular, $\langle\,.\,, z^m\rangle z^n \in \operatorname{clos}\mathcal{T}_1$. By linearity, $\langle\,.\,, p\rangle q \in \operatorname{clos}\mathcal{T}_1$ whenever p, q are polynomials. Because polynomials are dense in $A^2(\mathbf{D})$, necessarily $\langle\,.\,, f\rangle g \in \operatorname{clos}\mathcal{T}_1$ for all $f,\ g \in A^2(\mathbf{D})$, i.e. all one-dimensional operators are in $\operatorname{clos}\mathcal{T}_1$. Using the linearity of $\operatorname{clos}\mathcal{T}_1$ for the third time shows that all finite rank operators belong to $\operatorname{clos}\mathcal{T}_1$; since these are dense in $\mathtt{Comp}(A^2(\mathbf{D}))$, Theorem 2.4 follows. $\square$

Remark 2.8. For the Fock space $A^2(\mathbf{C}^N)$ in place of $A^2(\mathbf{D})$, the proof comes through essentially without alterations. The Taylor series (6) are convergent for all $x \in \mathbf{C}$, and so the formulas obtained from them are valid for all $t \in \mathbf{C}$ (instead of $|t| < 1 - |a|$). For the same reason, (9) becomes

$$\|F\|_2^2 = \sum_{n=0}^{\infty} n!2^n |F_n|^2,$$

and so $\|F\|_2 = \|\alpha\|_2$ this time, where $\alpha \in l^2$,

$$\alpha_k = (k!2^k)^{1/2}|F_k|,$$

and similarly for G. Formula (11) becomes

$$\begin{aligned} \|\alpha\|_2 . \left(\sum_{m=1}^{\infty} \frac{(M+m)!^2}{m!^2} \cdot \frac{t^{2m-1}}{(M+m)!.2^{M+m}} \right)^{1/2} &= \\ = \|\alpha\|_2 . \left(t \sum_{m=0}^{\infty} \frac{(M+m+1)!}{(m+1)!^2 \,.\, 2^{M+m+1}} t^{2m} \right)^{1/2} &= \\ = \|F\|_2 \,.\, t^{1/2} c_1(M,t), & \end{aligned}$$

where $c_1(M,t)$ tends to a finite limit as $t^2 \to 0+$, i.e. as $t \to 0$; the estimate (12) for σ_1

$$\sigma_1 \le \|F\|_2 . \|G\|_2 . t c_2(M,N,t)$$

follows. Similarly, (13) becomes

$$\|\beta\|_2 . \left(\sum_{n=2}^{\infty} \frac{(N+n)!}{n!^2} \, \frac{t^{2n-2}}{2^{N+n}} \right)^{1/2} = \|G\|_2 . t c_3(N,t),$$

with

$$c_3(N,t) = \left(\sum_{n=0}^{\infty} \frac{N+n+2)!}{(n+2)!^2} \, \frac{t^{2n}}{2^{N+n}} \right)^{1/2}$$

tending to a finite limit as $t \to 0$, and so

$$\sigma_2 \le c_4(M,N,t) . \|F\|_2 . \|G\|_2 . t$$

with $c_4(M,N,t)$ tending to a finite limit as $t \to 0$.

In the proof of Lemma 2.7, the functions $f_\delta \in \mathcal{D}(\mathbf{C})$ must be chosen so that

$$f_\delta(z) = 0 \qquad \text{if } |z-a| \ge \delta + \delta^2,$$

$$f_\delta(z) = 2\delta^{-2} \qquad \text{if } |z-a| \le \delta,$$

and $0 \le f_\delta(z) \le 2/\delta^2$ otherwise; here $0 < \delta < 1$. Proceed till you define ρ_1 and ρ_2 as in (14); then pass to polar coordinates to obtain

$$\rho_1 = \frac{2}{\delta^2}\int_0^\delta \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^\infty\sum_{m=0}^\infty f_n\overline{g_m} r^{n+m} e^{(n-m)it} e^{-r^2/2}\,dt\,r\,dr - f_0\overline{g_0} =$$
$$= \frac{1}{\delta^2}\sum_{n=0}^\infty 2^{n+1} f_n\overline{g_n}\int_0^{\delta^2/2} t^n e^{-t}\,dt - f_0\overline{g_0}.$$

For $n \ge 1$, the function $t^n e^{-t}$ increases on the interval $(0,n) \supset (0,\delta)$; consequently, $\int_0^{\delta^2/2} t^n e^{-t}\,dt \le (\delta^2/2)^{n+1} e^{-\delta^2/2}$, and

$$|\rho_1| \le \left|\frac{2}{\delta^2} f_0\overline{g_0}.(1-e^{-\delta^2/2}) - f_0\overline{g_0}\right| + \left|\frac{2}{\delta^2} e^{-\delta^2/2}\sum_{n=1}^\infty 2^{n+1} f_n\overline{g_n}\left(\frac{\delta^2}{2}\right)^{n+1}\right| =$$

$$= |f_0\overline{g_0}|.O\left(\frac{\delta^2}{2}\right) + 2e^{-\delta^2/2}.\sum_{n=1}^\infty |f_n\overline{g_n}|\delta^{2n}.$$

Employing the $\alpha, \beta \in l^2$ again, the last expression is seen to be bounded by

$$c_6(\delta).\delta^2.\|F\|_2.\|G\|_2,$$

with $c_6(\delta)$ tending to a finite limit as $\delta \to 0+$.

As for ρ_2, we have, again,

$$|\rho_2| \le \sup_{\delta<|z-a|<\delta+\delta^2} |f_\delta(z)f(z)\overline{g(z)}|\,.\int\limits_{\delta<|z-a|<\delta+\delta^2} dz \le$$
$$\le \frac{2}{\delta^2}\sup_{\delta<|z-a|<\delta+\delta^2}|f(z)g(z)|\,.\,\frac{\delta^3(\delta+2)}{2}\,e^{-\delta^2/2} \le$$
$$\le \delta(\delta+2)e^{-\delta^2/2}\sup_{\delta<|z-a|<\delta+\delta^2}|f(z)g(z)|,$$

$$\sup_{\delta<|z-a|<\delta+\delta^2}|f(z)g(z)| \le \|F\|_2.\|G\|_2.e^{(\delta+\delta^2)^2/2},$$

$$|\rho_2| \le \delta(\delta+2).\|F\|_2.\|G\|_2.e^{\delta^3(\delta+2)/2} = \delta.\|F\|_2.\|G\|_2.c_6(\delta),$$

where $c_6(\delta) \to 3$ as $\delta \to 0+$. Thus,

$$|\rho_1+\rho_2| \le \|F\|_2.\|G\|_2.\delta.c_7(\delta),$$

where $c_7(\delta) \to 3$ as $\delta \to 0+$, and Lemma 2.7 follows as before. This shows that the second proof of Theorem 2.4 works also for the Fock space $A^2(\mathbf{C}^N)$. □

Remark 2.9. Because $\texttt{Comp}(A^2(\Omega))$ is SOT-dense in $\mathcal{B}(A^2(\Omega))$ and norm convergence implies SOT-convergence, Theorem 2.4 yields another proof of Theorem 2.3. □

For $\Omega = \mathbf{D}$, there is a natural intermediate function space between $\mathcal{D}(\mathbf{D})$ and $L^\infty(\mathbf{D})$ — namely, $C(\overline{\mathbf{D}})$, the functions continuous on the closed unit disc $\overline{\mathbf{D}}$. The norm closure of the set $\mathcal{T}_2 = \{T_\phi : \phi \in C(\overline{\mathbf{D}})\}$ was described in my paper [16], using the techniques of Olin and Thomson and of Bunce.

Theorem 2.10. $\operatorname{clos} \mathcal{T}_2$ *is a* C^**-algebra and coincides with* $\mathcal{T}_2 + \mathtt{Comp}(A^2(\mathbf{D}))$.

Corollary 2.11. *For every* $T \in \operatorname{clos} \mathcal{T}_2$, $\sigma_e(T)$ *is connected. In particular,* $\operatorname{clos} \mathcal{T}_2 \neq \mathcal{B}(A^2(\mathbf{D}))$.

We omit the proofs, which can be found in [16]. Using our Theorem 2.4, they can also be deduced from results of Axler, Conway and McDonald [3].

Chapter 3. TOEPLITZ ALGEBRAS

As we have already mentioned, the results of the previous chapter prompt to ask whether the Toeplitz operators (e.g. on $A^2(\mathbf{D})$) are not actually norm-dense in the space of all linear operators; or if, at least, the C^*-algebra generated by them is not all of $\mathcal{B}(A^2(\mathbf{D}))$. In this chapter, we are going to prove that this is *not* the case: all Toeplitz operators on the Bergman space $A^2(\mathbf{D})$ are contained in a certain C^*-algebra which is a proper subalgebra of $\mathcal{B}(A^2(\mathbf{D}))$. The same result holds also for the classical Hardy space H^2, the Fock space $A^2(\mathbf{C})$, Bergman spaces $A^2(\Omega)$ for a wide class of domains $\Omega \subset \mathbf{C}$ and for spaces $H^2(\rho)$ (to be defined later in this chapter). Moreover, the corresponding algebras can be chosen to be, loosely speaking, *the same* (more precisely: spatially isomorphic to each other) in all these cases. Let us start with $A^2(\mathbf{D})$, where the proof is most transparent.

Define

$$\mathcal{A}(T_z) := \{T \in \mathcal{B}(A^2(\mathbf{D})) : T - T_z^* T T_z \in \mathtt{Comp}\}.$$

There is an alternative definition of $\mathcal{A}(T_z)$:

Proposition 3.1. $\mathcal{A}(T_z) = \{T \in \mathcal{B}(A^2(\mathbf{D})) : [T, T_z] \in \mathtt{Comp}\}$.

Proof. The operators

$$I - T_z^* T_z = \operatorname{diag}\left(1 - \frac{n+1}{n+2}\right) = \operatorname{diag}\left(\frac{1}{n+2}\right)$$

and

$$I - T_z T_z^* = \operatorname{diag}\left(1 - \frac{n}{n+1}\right) = \operatorname{diag}\left(\frac{1}{n+1}\right)$$

are compact; consequently,

$$T - T_z^* T T_z \in \mathtt{Comp} \implies T_z(T - T_z^* T T_z) = (T_z T - T T_z) + (I - T_z T_z^*) T T_z \in \mathtt{Comp} \iff$$

$$\iff T_z T - T T_z \in \mathtt{Comp},$$

and, on the other hand,

$$T_z T - T T_z \in \mathtt{Comp} \implies T_z^* T_z T - T_z^* T T_z = (T - T_z^* T T_z) - (I - T_z^* T_z) T \in \mathtt{Comp} \iff$$

$$\iff T - T_z^* T T_z \in \mathtt{Comp}.$$

□

The following theorem shows that $\mathcal{A}(T_z)$ is the algebra appealed to above.

Theorem 3.2. *(i) $\mathcal{A}(T_z)$ is a C^*-algebra.*
(ii) $\forall \phi \in L^\infty(\mathbf{D}) : T_\phi \in \mathcal{A}(T_z)$.

Proof. (i) It's clear that $\mathcal{A}(T_z)$ is a linear and selfadjoint set, which is moreover closed in the norm topology; so the only thing that remains to be checked is that it is closed under multiplication. But

$$[AB, T_z] = A(BT_z - T_z B) + (AT_z - T_z A)B = A[B, T_z] + [A, T_z]B,$$

which is compact if $[A, T_z]$ and $[B, T_z]$ are.

(ii) If $\phi \in L^\infty(\mathbf{D})$, then

$$T_\phi - T_z^* T_\phi T_z = T_\phi - T_{\overline{z}} T_\phi T_z = T_{\phi - \overline{z}\phi z} = T_{(1-|z|^2)\phi(z)}.$$

But $(1-|z|^2)\phi(z) \in V(\mathbf{D})$ and so the last operator is compact by Corollary 1.5. $\square$

Corollary 3.3. *The C^*-algebra generated by $\{T_\phi : \phi \in L^\infty(\mathbf{D})\}$ is strictly smaller than $\mathcal{B}(A^2(\mathbf{D}))$.*

Proof. In view of the preceding Theorem, it suffices to find an operator not in $\mathcal{A}(T_z)$; one of them is

$$J = \operatorname{diag}(-1)^n,$$

since

$$J - T_z^* J T_z = \operatorname{diag}((-1)^n - \frac{n+1}{n+2}(-1)^{n+1})$$

certainly is not compact. $\square$

Theorem 3.2 carries over trivially to the classical Hardy space H^2. Indeed, when T_ϕ is a Toeplitz operator on H^2, then

$$T_\phi = S^* T_\phi S,$$

where $S = T_z$ is the usual (forward) shift operator on H^2. Thus, if we define

$$\mathcal{A}(S) := \{T \in \mathcal{B}(H^2) : T - S^* T S \in \texttt{Comp}(H^2)\},$$

then the following assertions are immediate.

Proposition 3.4. $\mathcal{A}(S) = \{T \in \mathcal{B}(H^2) : [T, S] \in \texttt{Comp}(H^2)\}$.

Theorem 3.5. *(i) $\mathcal{A}(S)$ is a C^*-subalgebra of $\mathcal{B}(H^2)$.*
(ii) $T_\phi \in \mathcal{A}(S)$ for every Toeplitz operator T_ϕ on H^2.

Corollary 3.6. *The C^*-algebra generated by the Toeplitz operators in $\mathcal{B}(H^2)$ is strictly smaller than $\mathcal{B}(H^2)$.*

The proofs are similar to those for 3.1 — 3.3, and actually a lot simpler. In the Corollary, the same operator J (this time, of course, diagonality is understood with respect to the standard orthonormal basis $\{z^n\}_{n=0}^\infty$ of H^2) works. The algebras $\mathcal{A}(T_z)$ and $\mathcal{A}(S)$ are, in fact, isomorphic; moreover, the isomorphism $\mathcal{A}(T_z) \to \mathcal{A}(S)$ may be chosen to be spatial, i.e. of the form

$$T \mapsto W^* T W,$$

where W is a fixed unitary operator from H^2 onto $A^2(\mathbf{D})$. To see this, let W be the operator mapping the standard basis $\{z^n\}_{n \in \mathbf{N}}$ of H^2 onto the basis $\{\sqrt{n+1}\, z^n\}_{n \in \mathbf{N}}$ od $A^2(\mathbf{D})$,

$$W : \sum_{n=0}^\infty f_n z^n \mapsto \sum_{n=0}^\infty f_n \sqrt{n+1}\, z^n.$$

Then

$$\begin{aligned} T \in \mathcal{A}(T_z) &\iff [T, T_z] \in \texttt{Comp} \iff W^*TT_zW - W^*T_zTW \in \texttt{Comp} \iff \\ &\iff (W^*TW)(W^*T_zW) - (W^*T_zW)(W^*TW) \in \texttt{Comp} \iff \\ &\iff (W^*TW)S - S(W^*TW) \in \texttt{Comp} \iff W^*TW \in \mathcal{A}(S); \end{aligned}$$

here T_z is the Toeplitz operator on $A^2(\mathbf{D})$, not on H^2, and the last-but-one equivalence is due to the fact that

$$W^*T_zW - S = S.\operatorname{diag}(\sqrt{\frac{n+1}{n+2}} - 1)$$

is a compact operator[4].

In general, we may define

$$\mathcal{A}(M) := \{T \in \mathcal{B}(H) : [M, T] \in \texttt{Comp}(H)\}$$

for arbitrary operator M on a Hilbert space H. The following theorem generalizes the considerations of the previous paragraph.

Theorem 3.7. *(i) $\mathcal{A}(M) = \mathcal{A}(M + K)$ for arbitrary compact operator K.*

(ii) Suppose that $\sigma_e(M) = \mathbf{T}$ and $\operatorname{ind} M = -1$ (the Fredholm index). Then there exists a unitary operator $W : H^2 \to H$ such that the transformation

$$T \mapsto W^*TW$$

is a C^*-algebra isomorphism of $\mathcal{A}(M)$ onto $\mathcal{A}(S)$. In particular, $\mathcal{A}(M)$ is a proper C^*-subalgebra of $\mathcal{B}(H)$.

Proof. (i) is immediate (actually, it has already been used in the end of the last-but-one paragraph).

(ii) According to the Brown-Douglas-Fillmore theory [8], an operator M satisfying these conditions is unitarily equivalent to S modulo the compacts, i.e. there exists a unitary operator $W : H^2 \to H$ and a compact operator $K \in \texttt{Comp}(H)$ such that

$$WSW^* = M + K.$$

Owing to part (i), $\mathcal{A}(M) = \mathcal{A}(WSW^*)$, and rehearsing the argumentation of the last-but-one paragraph leads to the desired conclusion. □

Now we are in a position to prove the analogue of Theorem 3.2 for general Bergman spaces $A^2(\Omega)$. In case the domain Ω is bounded, a short proof may be given based on the results of Axler, Conway & McDonald [3] or of Olin & Thomson [23][5]. We shall present it first, and then, in case the domain Ω is simply connected, employ an idea of Axler to obtain a more elementary proof which uses only methods of complex function theory.

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[4]Here, once more, the diagonality is understood with respect to the standard basis of H^2.

[5]The main idea, however, goes back to Bunce[9].

We begin by recalling the pertinent results of Axler et al. [3]. Assume that Ω is bounded. A point $x \in \partial\Omega$ is called removable if there exists a neighbourhood V of x such that every function $f \in A^2(\Omega)$ can be analytically continued to V. (For instance, every isolated point of $\partial\Omega$ is removable, by a variant of Riemann's removable singularity theorem.) The collection of all removable boundary points is called $\partial_r\Omega$, the *removable boundary* of Ω; $\partial_e\Omega := \partial\Omega \setminus \partial_r\Omega$ is the *essential boundary* . It is proved in [3] that $\Omega \cup \partial_r\Omega$ is an open set and that $\partial_r\Omega$ has zero Lebesgue measure; consequently, $L^2(\Omega) = L^2(\Omega \cup \partial_r\Omega)$ and $A^2(\Omega) = A^2(\Omega \cup \partial_r\Omega)$. This makes it possible to replace, without loss of generality, Ω by $\Omega \cup \partial_r\Omega$, i.e. to assume that $\partial_r\Omega = \emptyset$, $\partial\Omega = \partial_e\Omega$. In that case, the following two assertions hold.

Proposition 3.8. (cf. [3], Prop. 8) *If*[6] $f \in C(\overline{\Omega})$, *then* $H_f \in \mathtt{Comp}(A^2(\Omega))$.

Proposition 3.9. ([3], Corol. 10) *If* $f \in C(\overline{\Omega})$, *then* $\sigma_e(T_f) = f(\partial\Omega)$.

Now we are ready to prove our main theorem.

Theorem 3.10. *Assume that Ω is a bounded domain in $\mathbf{C}$. Then there exists a unitary operator $W : H^2 \to A^2(\Omega)$ such that the transformation*

$$T \mapsto W^*TW, \qquad \mathcal{B}(A^2(\Omega)) \to \mathcal{B}(H^2),$$

sends every Toeplitz operator T_ϕ, $\phi \in L^\infty(\Omega)$, on $A^2(\Omega)$ to an element of $\mathcal{A}(S)$. In particular, the C^*-algebra generated by the Toeplitz operators on $A^2(\Omega)$ is a proper subalgebra of $\mathcal{B}(A^2(\Omega))$.

Proof. Without loss of generality we may assume $\mathbf{D} \subset \Omega$. Let $\Phi \in L^\infty(\Omega)$ be the function $z/|z|$ adjusted in a small neighbourhood of 0 so as to be continuous on $\overline{\Omega}$; for instance, take

$$\Phi(re^{it}) = \begin{cases} e^{it} & \text{if } r \geq 1, \\ re^{it} & \text{if } r \leq 1. \end{cases}$$

Because $\Phi \in C(\overline{\Omega})$, the Hankel operators H_Φ and $H_{\overline{\Phi}}$ are compact by Proposition 3.8 and, consequently, so must be the operator (cf. Proposition 1.2)

$$[T_\phi, T_\Phi] = H^*_{\overline{\Phi}} H_\phi - H^*_{\overline{\phi}} H_\Phi$$

for arbitrary $\phi \in L^\infty(\Omega)$; thus, $T_\phi \in \mathcal{A}(T_\Phi)$. In view of Proposition 3.9, $\sigma_e(T_\Phi) = \Phi(\partial\Omega) = \mathbf{T}$. If we prove that $\operatorname{ind} T_\Phi = -1$, we can apply Theorem 3.7 and the desired conclusion will follow.

For $0 \leq \theta \leq 1$, define

$$\Phi_\theta(re^{it}) = \begin{cases} r^\theta e^{it} & \text{for } r \geq 1, \\ re^{it} & \text{for } r \leq 1. \end{cases}$$

Then $\Phi_\theta \in L^\infty(\Omega)$, and so the operators T_{Φ_θ} are defined. Moreover, for arbitrary θ_1, $\theta_2 \in \langle 0, 1 \rangle$,

$$\|T_{\Phi_{\theta_1}} - T_{\Phi_{\theta_2}}\| \leq \|\Phi_{\theta_1} - \Phi_{\theta_2}\|_\infty = \sup_{1 \leq r \leq \operatorname{diam}\Omega} |r^{\theta_1} - r^{\theta_2}| \leq c.|\theta_1 - \theta_2|,$$

[6] $C(\overline{\Omega})$ is the space of functions continuous on the closure $\overline{\Omega}$ of Ω.

where diam Ω is the diameter of Ω and

$$c = \sup_{\substack{1 \le r \le \operatorname{diam}\Omega \\ 0 \le \theta \le 1}} |r^\theta \ln r|$$

does not depend on θ. This shows that the mapping $\theta \mapsto T_{\Phi_\theta}$ is continuous. Besides, $\sigma_e(T_{\Phi_\theta}) = \Phi_\theta(\partial_e \Omega) \not\ni 0$, i.e. all T_{Φ_θ} are Fredholm and so their index is defined. Since "ind" is a continuous integer-valued function, it must be constant along the path $\theta \mapsto T_{\Phi_\theta}$, whence $\operatorname{ind} T_{\Phi_{\theta_0}} = \operatorname{ind} T_{\Phi_{\theta_1}}$, or

$$\operatorname{ind} T_\Phi = \operatorname{ind} T_z .$$

But $\ker T_z = \{0\}$, while $\operatorname{Ran} T_z$ consists of all functions from $A^2(\Omega)$ that vanish at 0. Consequently, $\operatorname{ind} T_z = -1$, and the proof is complete. $\square$

If $\Omega \subset \mathbf{C}$ is simply connected, it turns out that the condition that Ω be bounded may be weakened — namely, it suffices that Ω have finite Lebesgue measure. (If the latter condition is not met, the spaces $A^2(\Omega)$ become too small — they won't even contain nonzero constant functions.) It is also not necessary to appeal to the results quoted above if one is willing to do a little computing. The subsequent three lemmas were inspired by the aforementioned work of Axler et al. [3].

Lemma 3.11. *Assume that $\Omega \subset \mathbf{C}$ is a domain of finite Lebesgue measure, $h \in H^\infty(\Omega)$, $1/h \notin H^\infty(\Omega)$, and* $\operatorname{Ran} T_h$ *is closed. Then there exists $N \in \mathbf{N}$ such that $h^{-N} \notin A^2(\Omega)$.*

Proof. The case $h = 0$ is trivial; otherwise, T_h is injective and, having closed range, must be bounded below, i.e. there exists $c > 0$ such that $\|T_h f\| \ge c\|f\| \quad \forall f \in A^2(\Omega)$. Let g be the characteristic function of the set $\{z \in \Omega : |h(z)| \le c/2\}$. Because $|\Omega| < +\infty$, g belongs to $L^2(\Omega)$. If $h^{-n} \in A^2(\Omega)$ for all $n = 0, 1, 2, \ldots$, then

$$|\langle g, \mathbf{1}\rangle| = |\langle g, T_{h^n} h^{-n}\rangle| \le \|T^*_{h^n} g\| . \|h^{-n}\|; \tag{17}$$

but

$$\|h^{-n}\| \le c^{-n} \|\mathbf{1}\|$$

and

$$\|T^*_{h^n} g\| \le \|\overline{h^n} g\| = \Big(\int_{|h(z)| \le c/2} |h(z)|^{2n} \, dz \Big)^{1/2} \le (c/2)^n \|g\|_2 ,$$

which when plugged into (17) yields

$$|\langle g, \mathbf{1}\rangle| \le 2^{-n} . \|g\|_2 . \|\mathbf{1}\|_2 .$$

It follows that $\langle g, \mathbf{1}\rangle = 0$, i.e. that $|h(z)| > c/2$ almost everywhere on Ω, which contradicts our assumption $1/h \notin H^\infty(\Omega)$. $\square$

Prior to stating the second lemma, let us recall some terminology and facts from the geometric function theory. For a simply connected domain $\Omega \subset \mathbf{C}$, denote $\overline{\Omega}$ and $\partial\Omega$ its closure and boundary in $\mathbf{G} := \mathbf{C} \cup \{\infty\}$, respectively. Consider the system $\mathcal{G}$

of all curves $\gamma : \langle 0,1\rangle \to \overline{\Omega}$ such that $\gamma(1) \in \partial\Omega$ an $\gamma(\langle 0,1)) \subset \Omega$. Two curves γ_0, $\gamma_1 \in \mathcal{G}$ are said to be equivalent if

(i) $\gamma_0(1) = \gamma_1(1) = z_0 \in \partial\Omega$

(ii) in every neighbourhood V of z_0, there exists a curve $\eta : \langle 0,1\rangle \to \Omega \cap V$ lying in $\Omega \cap V$ and such that $\eta(0)$ lies on γ_0 and $\eta(1)$ on γ_1.

This is readily seen to be indeed an equivalence relation; the equivalence classes are called *accessible boundary elements*[7]. The set of all accessible boundary elements of Ω will be denoted by $\mathfrak{B}$. Let $\Phi : \Omega \to \mathbf{D}$ be the Riemann mapping function[8]. The following theorem of Koebe can be found e.g. in Goluzin [18].

Theorem 3.12. *The map Φ may be extended to all accessible boundary elements by setting*

$$\Phi(\gamma) := \lim_{t\to 1-} \Phi(\gamma(t)),$$

where γ is an arbitrary curve from the equivalence class defining the accessible boundary element. This extended mapping (denoted also Φ) is injective on $\Omega \cup \mathfrak{B}$ and maps $\mathfrak{B}$ onto a dense subset of $\mathbf{T}$.

Now we may state the second and the third lemma.

Lemma 3.13. *Assume that $\Omega \subset \mathbf{C}$ is a simply connected domain of finite measure, $w \in \mathbf{T}$, and let $\Phi : \Omega \to \mathbf{D}$ be the Riemann mapping function. Then if the operator $T_{\Phi-w}$ is Fredholm, its index is -1.*

Proof. For arbitrary $x \in \mathbf{D}$, the operator $T_{\Phi-x}$ is injective and its range clearly consists exactly of functions vanishing at $\Phi^{-1}(x)$ (since $\Phi(z)-x$, loosely speaking, behaves like $z - \Phi^{-1}(x)$ in a sufficiently small neighbourhood of $\Phi^{-1}(x)$). Hence, $\operatorname{ind} T_{\Phi-x} = -1$ for $x \in \mathbf{D}$. Since ind is a continuous function where defined, the result follows upon letting $x \to w$. $\square$

Lemma 3.14. *Assume $\Omega \subset \mathbf{C}$ is a simply connected domain of finite measure and let $\overline{\Omega}$, $\partial\Omega$, $\mathfrak{B}$, Φ have the same meaning as above. Let $\lambda \in \mathfrak{B}$ be an accessible boundary element and $\mu = \Phi(\lambda) \in \mathbf{T}$. Then the operator $T_{\Phi-\mu}$ on $A^2(\Omega)$ is not Fredholm.*

Proof. Denote $h(z) = \Phi(z) - \mu$ (for brevity) and assume that, on the contrary, T_h is a Fredholm operator. Then $h \in H^\infty(\Omega)$, $1/h \notin H^\infty(\Omega)$ and $\operatorname{Ran} T_h$ is closed. According to Lemma 3.11, there exists $N \in \mathbf{N}$ such that $h^{-N} \in A^2(\Omega)$ but $h^{-N-1} \notin A^2(\Omega)$; in other words, $h^{-N} \in A^2(\Omega) \setminus \operatorname{Ran} T_h$. Choose $k \in A^2(\Omega) \ominus \operatorname{Ran} T_h$ satisfying $\langle h^{-N}, k\rangle = 1$. In view of the last lemma, $\operatorname{ind} T_h = -1$, and so $\operatorname{Ran} T_h = k^\perp$. Let f be an arbitrary function in $A^2(\Omega)$. Set $a_0 = \langle f, k\rangle$ and observe that

$$\langle f - a_0 h^{-N}, k\rangle = \langle f, k\rangle - \langle f, k\rangle\langle h^{-N}, k\rangle = 0, \tag{18}$$

which implies that $f - a_0 h^{-N} \in \operatorname{Ran} T_h$, i.e. $f - a_0 h^{-N} = h f_1$ for some $f_1 \in A^2(\Omega)$. Applying the same procedure to f_1 gives $f_2 \in A^2(\Omega)$ such that $f_1 - h^{-N} a_1 = h f_2$, where $a_1 = \langle f_1, k\rangle$; etc. etc. Thus, we arrive at an expansion

$$h^N f = a_0 + h a_1 + h^2 a_2 + \cdots + h^{n-1} a_{n-1} + h^{n+N} f_n.$$

[7]The usual definition of a boundary element reads somewhat differently, but for *accessible* boundary elements reduces to the one given here.

[8]Since we are dealing only with domains of finite measure, $\mathbf{G} \setminus \Omega$ contains at least two points.

We claim that, letting $n \to +\infty$, the resulting series on the right-hand side will actually converge on some set; this can be seen as follows. Since T_h is Fredholm and injective, it must be bounded below — say, $\|T_h f\| \geq c\|f\| \quad \forall f \in A^2(\Omega)$ for some $c > 0$. Because $a_{n-1} = \langle f_{n-1}, k\rangle$,

$$\|f_n\| \leq \frac{1}{c}\|hf_n\| \leq \frac{1}{c}(\|f_{n-1}\|+\|h^{-N}a_{n-1}\|) \leq \frac{1}{c}\|f_{n-1}\|.(1+\|h^{-N}\| \, . \, \|k\|) := \|f_{n-1}\|/\delta,$$

where $\delta = \delta(h) > 0$ is independent of n and f. Iterating the last inequality gives $\|f_n\| \leq \delta^{-n}\|f\|$. Consequently[9], $|f_n(z)| \leq \|f_n\|\|g_z\| \leq \|f\|.\|g_z\|/\delta^n$ and so

$$|h^n(z)f_n(z)| \leq 2^{-n}.\|g_z\|.\|f\|$$

whenever $|h(z)| < \delta/2$. It follows that we indeed may let $n \to +\infty$ in (18) to obtain

$$h(z)^N f(z) = \sum_{n=0}^{\infty} a_n h(z)^n,$$

the series converging on the (open) set $\{z \in \Omega : |h(z)| < \delta/2\}$.

Now, apply the mapping Φ — let $y = \Phi(z)$. The last mentioned set is mapped bijectively onto

$$\mathcal{W} = \{y \in \mathbf{D} : |y - \mu| < \delta/2\}.$$

Thus, we have proved that for an arbitrary function $f \in A^2(\Omega)$ the function $(y - \mu)^N f\Phi^{-1}(y)$ admits the expansion

$$(y-\mu)^N.f\Phi^{-1}(y) = \sum_{n=0}^{\infty} a_n.(y-\mu)^n$$

which converges for $y \in \mathcal{W}$. But this means that the radius of convergence of the series on the right-hand side must be at least $\delta/2$; consequently, the function on the left-hand side admits an analytic continuation to the disc $\{y : |y-\mu| < \delta/2\}$. Because Ω is assumed to have finite measure, $A^2(\Omega) \supset H^\infty(\Omega)$, and so the above conclusion applies, in particular, to every function $f \in H^\infty(\Omega)$. But the mapping $f \mapsto f\Phi^{-1}$ is obviously a bijection between $H^\infty(\Omega)$ and $H^\infty(\mathbf{D})$; hence, our conclusion implies that for arbitrary function $g \in H^\infty(\mathbf{D})$ the function

$$(y-\mu)^N.g(y)$$

can be analytically continued to the disc $\{y : |y - \mu| < \delta/2\}$ for some $\delta > 0$, i.e. has at most a pole of order N at the point $y = \mu$. This is clearly a contradiction, and the proof is complete. □

Remark 3.15. So far, we have used Koebe's theorem only at the very beginning of the proof of Lemma 3.14 — to deduce that $1/h \notin H^\infty(\Omega)$. It's quite probable that

[9]Recall that g_z is the evaluation functional at $z \in \Omega$, which is always bounded in view of Proposition 1.1.

a more elementary direct proof may be found, avoiding the use of Koebe's theorem; we won't, however, pursue this matter further. $\square$

Corollary 3.16. *Let $\Omega \subset \mathbf{C}$ be a simply connected domain of finite measure and $\Phi : \Omega \to \mathbf{D}$ the Riemann mapping function. Then $\sigma_e(T_\Phi) = \mathbf{T}$.*

Proof. In course of the proof of Lemma 3.13, it was shown that $T_{\Phi-x}$ is Fredholm (with index -1) whenever $x \in \mathbf{D}$. When $x \notin \overline{\mathbf{D}}$, $\Phi(z) - x$ is invertible in $H^\infty(\Omega)$ and so $T_{\Phi-x}$ is invertible. It follows that $\sigma_e(T_\Phi) \subset \mathbf{T}$. By the last lemma, $\Phi(\lambda) \in \sigma_e(T_\Phi)$ whenever λ is an accessible boundary element of Ω; according to Koebe's theorem (Theorem 3.12), such points $\Phi(\lambda)$ form a dense subset of $\mathbf{T}$. It follows that $\sigma_e(T_\Phi) = \mathbf{T}$ as asserted. $\square$

Theorem 3.17. *Assume $\Omega \subset \mathbf{C}$ is a simply connected domain of finite Lebesgue measure and let $\Phi : \Omega \to \mathbf{D}$ be the Riemann mapping function. Then $T_\phi \in \mathcal{A}(T_\Phi)$ $\forall \phi \in L^\infty(\Omega)$ and there exists a unitary operator $W : H^2 \to A^2(\Omega)$ such that the transformation $T \mapsto W^*TW$ establishes a C^*-isomorphism of $\mathcal{A}(T_\Phi)$ onto $\mathcal{A}(S)$. In particular, the C^*-algebra generated by the Toeplitz operators T_ϕ, $\phi \in L^\infty(\Omega)$, is a proper subalgebra of $\mathcal{B}(A^2(\Omega))$.*

Proof. Let $\phi \in L^\infty(\Omega)$. Since $\Phi(z) \to \mathbf{T}$ as $z \in \Omega$ approaches $\partial\Omega \subset \mathbf{G}$, there exists a compact set $K \subset \Omega$ such that $|\phi|.(1 - |\Phi|^2) < \epsilon$ on $\Omega \setminus K$. Let χ be the characteristic function of K. According to Proposition 1.4, the Toeplitz operator with symbol $\chi\phi.(1 - |\Phi|^2)$ is compact; on the other hand, the norm of the Toeplitz operator with symbol $(1-\chi)\phi(1-|\Phi|^2)$ does not exceed ϵ. Since $\epsilon > 0$ was arbitrary, $T_{(1-|\Phi|^2)\phi} \in \mathtt{Comp}$. Hence

$$T_\phi - T_\Phi^* T_\phi T_\Phi \in \mathtt{Comp} \qquad \forall \phi \in L^\infty(\Omega). \tag{19}$$

Owing to Corollary 3.16, $\sigma_e(T_\Phi) = \mathbf{T}$; in course of the proof of Lemma 3.13, T_Φ was shown to be Fredholm with index -1. By the Brown-Douglas-Fillmore theory [8], T_Φ is unitarily equivalent modulo compacts to the shift operator S (on H^2); hence $I - T_\Phi^* T_\Phi$ must be compact, and multiplying (19) by T_Φ from the left yields (cf. the proof of Proposition 3.1)

$$T_\Phi T_\phi - T_\phi T_\Phi \in \mathtt{Comp},$$

i.e. $T_\phi \in \mathcal{A}(T_\Phi)$. It remains only to make use of Theorem 3.7.

The Theorems 3.2, 3.5, 3.10 & 3.17 imply that the same situation is encountered for $A^2(\mathbf{D})$, H^2 and $A^2(\Omega)$ for Ω bounded (not necessarily simply connected) or for Ω simply connected and of finite measure. Prior to investigating what happens for the Fock space, we briefly discuss another class of spaces which also seem to join here.

Let ρ be a positive Borel measure on $\langle 0, 1\rangle$. Set

$$a_n = \int_{\langle 0,1\rangle} t^n \, d\rho(t)$$

and define $H^2(\rho)$ to be the space of all functions $f(z) = \sum_{n=0}^\infty f_n z^n$ analytic on $\mathbf{D}$ for which

$$\|f\|_\rho := \left(\sum_{n=0}^\infty a_n |f_n|^2 \right)^{1/2} < +\infty.$$

Example: When $d\rho(r) = dr$ (the Lebesgue measure on $\langle 0,1\rangle$), $H^2(\rho)$ is our old friend $A^2(\mathbf{D})$.

The following propositions summarize basic properties of these spaces.

Proposition 3.18. *(i) $H^2(\rho)$ is a Hilbert space with scalar product*

$$\langle f,g\rangle_\rho := \sum_{n=0}^{\infty} a_n f_n \overline{g_n}, \qquad \text{where } f(z)=\sum_{n=0}^{\infty} f_n z^n,\ g(z)=\sum_{n=0}^{\infty} g_n z^n.$$

The set $\{z^n/\sqrt{a_n}\}_{n\in\mathbf{N}}$ is an orthonormal basis for $H^2(\rho)$.

(ii) If $\rho(\{1\})=0$ (i.e. ρ has no mass at 1), then

$$\|f\|_\rho^2 = \int_{\langle 0,1\rangle}\int_0^{2\pi} |f(r^{1/2}e^{it})|^2 \, \frac{dt}{2\pi}\, d\rho(r)$$

for any function f analytic on $\mathbf{D}$. If $\rho(\{1\})>0$, the last formula still holds if we agree to set

$$\int_0^{2\pi} |f(e^{it})|^2\,\frac{dt}{2\pi} = \sup_{0<r<1}\int_0^{2\pi}|f(re^{it})|^2\,\frac{dt}{2\pi}.$$

Proof. (i) is immediate; in fact, the mapping defined by $f\mapsto\{\sqrt{a_n}f_n\}_{n\in\mathbf{N}}$ is an isometric isomorphism of $H^2(\rho)$ onto $l^2(\mathbf{N})$.
(ii) Pass to the polar coordinates and compute to get

$$\int_{\langle 0,R)}\int_0^{2\pi}|f(r^{1/2}e^{it})|^2\,\frac{dt}{2\pi}\,d\rho(r) = \sum_{n=0}^{\infty}\left(\int_{\langle 0,R)} r^n\,d\rho(r)\right).|f_n|^2.$$

Letting $R\nearrow 1$ gives the desired formula. (Interchanges of integration and summation signs can be justified by the uniform convergence of $\sum_{n=0}^{\infty} f_n z^n$ on compact subsets of $\mathbf{D}$, and by the Lebesgue monotone convergence theorem.) $\square$

Proposition 3.19. *The evaluation functionals $f\mapsto f(\lambda)$, $\lambda\in\mathbf{D}$, on $H^2(\rho)$ are continuous for all $\lambda\in\mathbf{D}$ if and only if $1\in\operatorname{supp}\rho$.*

Proof. Since $f(\lambda)=\sum_{n=0}^{\infty} f_n\lambda^n = \sum_{n=0}^{\infty} a_n f_n\overline{\left(\frac{\overline{\lambda}^n}{a_n}\right)}$, the evaluation functional at $\lambda\in\mathbf{D}$ is continuous on $H^2(\rho)$ iff the function

$$g_\lambda(z)=\sum_{n=0}^{\infty}\frac{\overline{\lambda}^n z^n}{a_n}$$

belongs to $H^2(\rho)$. Because $\|g_\lambda\|_\rho^2=\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{a_n}$, this happens iff the radius of convergence of the series $\sum_{n=0}^{\infty}\frac{x^n}{a_n}$ is at least one, i.e. iff

$$\limsup |a_n|^{1/n}\geq 1.$$

The expression $a_n^{1/n} = \left(\int_{\langle 0,1\rangle} t^n \, d\rho(t)\right)^{1/n}$ is the norm of the function $t \mapsto t$ in the Lebesgue space $L^n(\langle 0,1\rangle, d\rho)$. For ρ finite (which is our case) this is well-known to tend to the norm in $L^\infty(\langle 0,1\rangle, d\rho)$, i.e. to $\operatorname{ess\,sup} t = \sup\{t : t \in \operatorname{supp}\rho\}$. This is ≥ 1 if and only if $1 \in \operatorname{supp}\rho$. $\square$

In order to define Toeplitz operators on $H^2(\rho)$, we need it to be a subspace of "some L^2". At that moment, technical difficulties arise when ρ has positive mass at 1, i.e. when $\rho(\{1\}) > 0$; so let us agree to exclude this case. Also, replacing ρ by $c\rho$ for some number $c > 0$ leads to isomorphic spaces $H^2(\rho)$ and $H^2(c\rho)$; so there is no harm in assuming $\rho(\langle 0,1\rangle) = 1$. Thus we are lead to the condition

$$\rho(\{1\}) = 0, \qquad \rho(\langle 0,1\rangle) = 1, \qquad 1 \in \operatorname{supp}\rho, \tag{20}$$

whose validity shall be assumed from now on. Proposition 3.18(ii) then implies that $H^2(\rho)$ is a closed subspace of the Hilbert space

$$L^2(\rho) := \left\{ f \text{ on } \mathbf{D} : \int_{\langle 0,1\rangle} \int_0^{2\pi} |f(r^{1/2}e^{it})|^2 \, \frac{dt}{2\pi} \, d\rho(r) \right\} < +\infty$$

(with the obvious inner product). Let P_+ be the orthogonal projection of $L^2(\rho)$ onto $H^2(\rho)$ and define Toeplitz operator $T_\phi : H^2(\rho) \to H^2(\rho)$ by

$$T_\phi f := P_+ \phi f, \qquad f \in H^2(\rho),$$

where $\phi \in L^\infty(\rho) := L^\infty(\mathbf{D}, dt\, d\rho(r))$. The formulas from Proposition 1.2

$$T_\phi T_g = T_{\phi g}, \quad T_{\overline{g}} T_\phi = T_{\overline{g}\phi}, \qquad \text{for } \phi \in L^\infty(\rho), g \in H^\infty(\mathbf{D}) \subset L^\infty(\rho),$$

are readily seen to remain in force, as well as Corollary 1.5. (The proofs carry over without change.) Consequently,

$$T_\phi - T_z^* T_\phi T_z = T_{(1-|z|^2)\phi} \in \mathtt{Comp}(H^2(\rho)). \tag{21}$$

The following proposition implies that this is equivalent to $T_\phi \in \mathcal{A}(T_z)$ and that $\mathcal{A}(T_z)$ is, once again, C^*-isomorphic to $\mathcal{A}(S)$.

Proposition 3.20. *Assume that the condition (20) is fulfilled. Let Z be the (forward) shift operator on $H^2(\rho)$ with respect to the orthonormal basis $\{z^n/\sqrt{a_n}\}_{n\in\mathbf{N}}$. Then T_z is a compact perturbation of Z.*

Proof. Denote, for a little while, $e_n = z^n/\sqrt{a_n}$, so that $Ze_n = e_{n+1}$ ($n \in \mathbf{N}$). By definition, $T_z z^n = z^{n+1}$, i.e. $T_z e^n = \sqrt{\dfrac{a_{n+1}}{a_n}}\, e_{n+1}$ and $Z - T_z = Z.\operatorname{diag}\left(1 - \sqrt{\dfrac{a_{n+1}}{a_n}}\right)$. Hence it suffices to show that $\dfrac{a_{n+1}}{a_n} \to 1$.

Clearly $a_{n+1} \leq a_n$ and, by Hölder's inequality, $a_n^{1/n} \leq a_m^{1/m}$ if $m > n$. Taking $m = n+1$ gives

$$a_n^{\frac{n+1}{n}} \leq a_{n+1} \leq a_n,$$

or

$$a_n^{1/n} \le \frac{a_{n+1}}{a_n} \le 1.$$

In course of the proof of Proposition 3.19, the left-hand side was observed to tend to $\sup(\operatorname{supp}\mu) = 1$ as $n \to \infty$. The proposition follows. $\square$

Corollary 3.21. *Assume that (20) is fulfilled. Then $T_\phi \in \mathcal{A}(T_z) \quad \forall \phi \in L^\infty(\rho)$, and $\mathcal{A}(T_z) = \mathcal{A}(Z)$ is spatially isomorphic to $\mathcal{A}(S)$.*

Proof. Premultiplying (21) by T_z shows that

$$T_z T_\phi - T_z T_z^* T_\phi T_z = [T_z, T_\phi] + (I - T_z T_z^*) T_\phi T_z$$

is compact; owing to the last proposition, $I - T_z T_z^* \in \mathtt{Comp}$, and so also $[T_z, T_\phi] \in \mathtt{Comp}$, i.e. $T_\phi \in \mathcal{A}(T_z)$. By Theorem 3.7(i), $\mathcal{A}(T_z) = \mathcal{A}(Z)$. Finally, if we define $W : H^2 \to H^2(\rho)$ by $z^n \in H^2 \mapsto z^n/\sqrt{a_n} \in H^2(\rho)$, then $Z = WSW^*$, and so the transformation $T \in \mathcal{A}(Z) \mapsto W^*TW \in \mathcal{A}(S)$ is a C^*-algebra isomorphism of $\mathcal{A}(Z)$ onto $\mathcal{A}(S)$. $\square$

Corollary 3.22. *Assume that (20) is fulfilled. Then the C^*-algebra generated by T_ϕ, $\phi \in L^\infty(\rho)$, is a proper subalgebra of $\mathcal{B}(H^2(\rho))$.*

Remark 3.23. We conclude our brief excursion into $H^2(\rho)$ spaces by a simple example. Take ρ to be the Lebesgue measure on $\langle 0, 1\rangle$. Then

$$a_n = \frac{1}{n+1}, \qquad n \in \mathbf{N},$$

$H^2(\rho)$ is but our old friend $A^2(\mathbf{D})$, and the last Corollary reduces to Theorem 3.2. $\square$

Now let us turn our attention to the Fock space $A^2(\mathbf{C})$. Recall that it has an orthonormal basis $\{e_n\}_{n=0}^\infty$,

$$e_n(z) := (n!\, 2^n)^{-1/2}\, z^n.$$

Denote Z the forward shift operator with respect to this basis, and let

$$\Phi(z) = \frac{z}{|z|} = e^{i \arg z}.$$

Theorem 3.24. *(i) The operator T_Φ is a compact perturbation of Z. Consequently, $\mathcal{A}(T_\Phi) = \mathcal{A}(Z)$.*

(ii) $T_f \in \mathcal{A}(T_\Phi)$, i.e. $T_f T_\Phi - T_\Phi T_f \in \mathtt{Comp}$, for every $f \in L^\infty(\mathbf{C})$.

(iii) There exists a unitary operator $W : H^2 \to A^2(\mathbf{C})$ such that the transformation $T \mapsto W^*TW$ is a C^*-isomorphism of $\mathcal{A}(Z)$ onto $\mathcal{A}(S)$.

In particular, the C^*-algebra generated by all T_f, $f \in L^\infty(\mathbf{C})$, is a proper subset of $\mathcal{B}(A^2(\mathbf{C}))$.

Proof. (i) Compute:

$$\langle T_\Phi z^n, z^m\rangle = \int_{\mathbf{C}} \frac{z}{|z|}\, z^n \overline{z}^m \, d\mu(z) = \int_0^{+\infty} \int_0^{2\pi} r^{n+m} e^{(n-m+1)it}\, \frac{dt}{2\pi}\, e^{-r^2/2} r\, dr.$$

This is zero unless $m = n+1$, and in that case it equals

$$\int_0^{+\infty} r^{2n+1} e^{-r^2/2} r\, dr = \int_0^{+\infty} 2^{n+\frac{1}{2}} t^{n+\frac{1}{2}} e^{-t}\, dt = 2^{n+\frac{1}{2}} \Gamma(n+\frac{3}{2}),$$

where Γ is Euler's gamma-function. Thus

$$\langle T_\Phi e_n, e_m\rangle = \begin{cases} 0 & \text{if } m \neq n+1 \\ (n!\, 2^n)^{-1/2} . (m!\, 2^m)^{-1/2} . 2^{n+\frac{1}{2}} \Gamma(n+\frac{3}{2}) & \text{if } m = n+1. \end{cases}$$

Consequently, $T_\Phi e_n = c_n e_{n+1}$, where

$$c_n = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)^{1/2} . \Gamma(n+2)^{1/2}}.$$

It follows that

$$Z - T_\Phi = Z \,.\, \mathrm{diag}(1 - c_n),$$

and in order to verify our claim it suffices to show that $c_n \to 1$ as $n \to +\infty$. According to Stirling's formula,

$$\Gamma(x+1) \sim \sqrt{2\pi}\, x^{x+\frac{1}{2}}\, e^{-x},$$

where "$\sim$" means that the ratio of the right-hand to the left-hand side approaches 1 as $x \to +\infty$. Substituting this into the expression for c_n produces

$$c_n \sim \frac{(n+\frac{1}{2})^{n+1} \,.\, e^{-n-\frac{1}{2}} \,.\, \sqrt{2\pi}}{n^{\frac{n}{2}+\frac{1}{4}} \,.\, e^{-\frac{n}{2}} \,.\, \sqrt[4]{2\pi} \,.\, (n+1)^{\frac{n}{2}+\frac{3}{4}} . e^{-\frac{n}{2}-\frac{1}{2}} . \sqrt[4]{2\pi}}.$$

The terms containing π cancel, as well as those containing e, and what remains is the product of

$$\left(\frac{n+\frac{1}{2}}{n}\right)^{n/2}, \qquad \left(\frac{n+\frac{1}{2}}{n+1}\right)^{\frac{n+1}{2}} \qquad \text{and} \qquad \frac{(n+\frac{1}{2})^{1/2}}{n^{1/4} . (n+1)^{1/4}},$$

which tend to $e^{1/4}$, $e^{-1/4}$ and 1, respectively. So, indeed, $c_n \to 1$ and the assertion follows.

(ii) Recall the formulas (cf. Proposition 1.6)

$$T_{fg} - T_f T_g = H^*_{\overline{f}} H_g, \qquad T_f T_g - T_g T_f = H^*_{\overline{g}} H_f - H^*_{\overline{f}} H_g, \tag{22}$$

which hold for arbitrary $f, g \in L^\infty(\mathbf{C})$. Owing to the second one,

$$T_f T_\Phi - T_\Phi T_f = H^*_{\overline{\Phi}} H_f - H^*_{\overline{f}} H_\Phi$$

will be compact for arbitrary $f \in L^\infty(\mathbf{C})$ if H_Φ, $H_{\overline{\Phi}} \in \mathtt{Comp}$. The latter is equivalent to $H_\Phi^* H_\Phi$, $H_{\overline{\Phi}}^* H_{\overline{\Phi}} \in \mathtt{Comp}$, respectively, and the first formula in (22) shows that this in turn is equivalent to

$$I - T_\Phi^* T_\Phi \qquad \text{and} \qquad I - T_\Phi T_\Phi^* \in \mathtt{Comp},$$

respectively. Owing to part (i), the last two operators are compact perturbations of $I - Z^*Z = 0$ and $I - ZZ^* = \langle\,.\,, e_0\rangle e_0$, respectively, and the result follows.

(iii) Define $W : H^2 \to A^2(\mathbf{C})$ by mapping the standard basis of H^2 onto the basis $\{e_n\}_{n\in\mathbf{N}}$ of $A^2(\mathbf{C})$,

$$W:\ z^n \in H^2 \mapsto \frac{z^n}{\sqrt{n!\,2^n}} \in A^2(\mathbf{C}).$$

This operator is unitary and the transformation $T \mapsto W^*TW$ maps Z to S; hence, as before, it induces a C^*-isomorphism of $\mathcal{A}(Z) = \mathcal{A}(T_\Phi)$ onto $\mathcal{A}(S)$. The proof is complete. $\square$

A variant of this result may also be obtained for $A^2(\mathbf{C}^N)$, $N \geq 2$; however, things come off a little different this time — the corresponding C^*-algebra is no longer spatially isomorphic to $\mathcal{A}(S)$. All the same, it is still a proper subset of $\mathcal{B}(A^2(\mathbf{C}^N))$.

We shall need some results of Berger and Coburn [6]. Define

$$ESV := \{\Phi \in L^\infty(\mathbf{C}^N) : \text{ess}\lim_{|z|\to+\infty} \sup_{|z-w|\leq 1} |\Phi(z) - \Phi(w)| = 0\}$$

and

$$BCESV := \{\Phi \in ESV : \Phi \text{ is continuous on } \mathbf{C}^N\}.$$

Here, as usual,

$$|x| = \left(\sum_{n=1}^N |x_n|^2\right)^{1/2} \qquad \text{for } x = (x_1, x_2, \ldots, x_N) \in \mathbf{C}^N.$$

Further, let $\mathcal{S} := \{x \in \mathbf{C}^N : |x| = 1\}$ be the unit sphere in $\mathbf{C}^N$.

Proposition 3.25. *Let $G : \mathcal{S} \to \mathbf{C}$ be a continuous function on $\mathcal{S}$. Define*

$$\Phi(rx) = \begin{cases} G(x) & \text{if } r \geq 1, \\ rG(x) & \text{if } r \leq 1, \end{cases} \qquad x \in \mathcal{S},\ 0 \leq r < +\infty. \tag{23}$$

Then
(i) $\Phi \in BCESV$, *and*
(ii) *the Hankel operators* H_Φ, $H_{\overline{\Phi}}$ *are compact.*
Assume further that

$$G(\mathcal{S}) = \mathbf{T}. \tag{24}$$

Then also
(iii) $\sigma_e(T_\Phi) = \mathbf{T}$ *and*

(iv) $\operatorname{ind} T_\Phi = 0$ *if* $N \geq 2$*, and* $\operatorname{ind} T_\Phi$ *is minus the winding number of the function* $G : \mathbf{T} \to \mathbf{T}$ *(with respect to the origin) when* $N = 1$*.*

Proof. (i) Φ is continuous and bounded since G is, and $\Phi \in ESV$ in view of [6], Theorem 3(i).

(ii) Theorem 11 of [6] says that H_Φ and $H_{\overline{\Phi}}$ are compact for arbitrary $\Phi \in ESV$.

(iii) & (iv) Immediate consequences of [6], Theorem 19. □

Remark 3.26. It is possible to prove part (iv) in another way, using the idea from the end of the proof of Theorem 3.10. Suppose that $\theta \mapsto G_\theta$, $\theta \in \langle 0, 1\rangle$, $G_\theta \in C(\mathcal{S})$, is a homotopy between G_0 and G_1; construct functions Φ_θ according to (23) and consider the Toeplitz operators T_{Φ_θ}. It can be shown that T_{Φ_θ} are Fredholm operators $\forall \theta \in \langle 0, 1\rangle$, and, consequently, $\operatorname{ind} T_{\Phi_0} = \operatorname{ind} T_{\Phi_1}$. If $N \geq 2$, the homotopy group $\pi(\mathcal{S}, \mathbf{T}) = \pi_{2N-1}(\mathbf{T})$ is trivial; hence there is a homotopy connecting $G_0 = G$ to $G_1 = \mathbf{1}$. It follows that

$$\operatorname{ind} T_\Phi = \operatorname{ind} T_{\mathbf{1}} = \operatorname{ind} I = 0.$$

If $N = 1$, $\pi(\mathbf{T}, \mathbf{T}) = \pi_1(\mathbf{T})$ is isomorphic to $\mathbf{Z}$; an isomorphism is given by $G \mapsto \operatorname{wind} G$. It follows that there is a homotopy connecting $G_0 = G$ to G_1, $G_1(e^{it}) = e^{kit}$, $k = \operatorname{wind} G$, and

$$\operatorname{ind} T_\Phi = \operatorname{ind} T_{\Phi_1} = -k.$$

Thus the occurence of two cases — $N = 1$ versus $N \geq 2$ — in the part (iv) is of topological nature, being related to (non)vanishing of the homotopy groups $\pi_n(\mathbf{T})$. □

Theorem 3.27. *Assume that the functions* $G : \mathcal{S} \to \mathbf{C}$ *and* $\Phi : \mathbf{C}^N \to \mathbf{C}$ *satisfy the conditions (23),(24) and that* $N \geq 2$*. Then*

(a) $T_f \in \mathcal{A}(T_\Phi)$ *for all* $f \in L^\infty(\mathbf{C}^N)$*.*

(b) There exists a unitary operator $W : L^2(\mathbf{T}) \to A^2(\mathbf{C}^N)$ *such that the transformation* $T \mapsto W^*TW$ *is a* C^**-isomorphism of* $\mathcal{A}(T_\Phi)$ *onto* $\mathcal{A}(U)$*, where* U *is the bilateral (forward) shift operator (the multiplication by* z*) on the Lebesgue space* $L^2(\mathbf{T})$*. In particular,* $\mathcal{A}(T_\Phi)$ *is a* C^**-algebra.*

(c) The operator $J : L^2(\mathbf{T}) \to L^2(\mathbf{T})$*,* $Jf(z) := f(-z)$*, does not belong to* $\mathcal{A}(U)$*. Consequently,* $\mathcal{A}(T_\Phi)$ *is a proper* C^**-subalgebra of* $\mathcal{B}(A^2(\mathbf{C}^N))$*.*

Proof. (a) For arbitrary $f \in L^\infty(\mathbf{C}^N)$,

$$T_f T_\Phi - T_\Phi T_f = H^*_{\overline{\Phi}} H_f - H^*_{\overline{f}} H_\Phi \qquad \text{(cf. Proposition 1.6)},$$

and the operators H_Φ, $H_{\overline{\Phi}}$ are compact by Theorem 3.24(ii).

(b) According to Theorem 3.24(iii) & (iv), $\sigma_e(T_\Phi) = \mathbf{T} = \sigma_e(U)$ and $\operatorname{ind} T_\Phi = 0 = \operatorname{ind} U$. Hence, by the Brown-Douglas-Fillmore theory [8], there exists a unitary operator $W : L^2(\mathbf{T}) \to A^2(\mathbf{C}^N)$ such that

$$W^* T_\Phi W = U + K,$$

where $K \in \texttt{Comp}$. The result follows in the same way as in the proof of Theorem 3.7, with S replaced by U.

(c) With respect to the standart orthonormal basis $\{e_n\}_{n\in\mathbf{Z}}$, $e_n(z)=z^n$, $z\in\mathbf{T}$, of $L^2(\mathbf{T})$, the operators J and U are given by

$$Ue_n = e_{n+1}, \qquad Je_n = (-1)^n e_n \qquad (n\in\mathbf{Z}).$$

It follows that $UJ-JU=2UJ$; but the operator UJ is unitary, and so certainly not compact. $\square$

Remark 3.28. To be precise, we ought to check that there exist functions G and Φ satisfying the conditions (23) and (24). As an example, take $G(z)=e^{4i\mathrm{Re}\, z_1}$. $\square$

The argument above applies also in the case $N=1$; one has only to replace $L^2(\mathbf{T})$ by H^2 and U by S^k or $S^{*(-k)}$ when $k=-\operatorname{ind} T_\Phi=\operatorname{wind} G\neq 0$. In particular, if $G:\mathbf{T}\to\mathbf{T}$ is the identity, we get another proof of Theorem 3.24.

What's the relationship between $\mathcal{A}(S)$ and $\mathcal{A}(U)$? Since H^2 is a subspace of $L^2(\mathbf{T})$, we may consider $\mathcal{B}(H^2)$ to be a subset (in fact, a C^*-subalgebra) of $\mathcal{B}(L^2(\mathbf{T}))$ by identifying $T\in\mathcal{B}(H^2)$ with $\begin{pmatrix} T & 0\\ 0 & 0\end{pmatrix}\in\mathcal{B}(H^2\oplus H^2_-)$, $H^2_-:=L^2(\mathbf{T})\ominus H^2$. The mapping $A\mapsto P_+A\restriction H^2$, i.e.

$$\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\in\mathcal{B}(H^2\oplus H^2_-)\longmapsto A_{11}\in\mathcal{B}(H^2),$$

is then a projection of $\mathcal{B}(L^2(\mathbf{T}))$ onto $\mathcal{B}(H^2)$.

Theorem 3.29. *Under this identification, $\mathcal{A}(S)$ becomes $\mathcal{A}(U)\cap\mathcal{B}(H^2)$. Moreover, $P_+A\restriction H^2\in\mathcal{A}(S)$ whenever $A\in\mathcal{A}(U)$.*

Proof. With respect to the decomposition $L^2(\mathbf{T})=H^2\oplus H^2_-$,

$$U=\begin{pmatrix} S & K_1\\ 0 & *\end{pmatrix},$$

where $K=\langle\,.\,,\overline{z}\rangle\mathbf{1}$ is a compact operator. If $T\in\mathcal{B}(H^2)$, then

$$\begin{pmatrix} T & 0\\ 0 & 0\end{pmatrix}U-U\begin{pmatrix} T & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix} TS & TK_1\\ 0 & 0\end{pmatrix}-\begin{pmatrix} ST & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix} [T,S] & TK_1\\ 0 & 0\end{pmatrix},$$

and so $\begin{pmatrix} T & 0\\ 0 & 0\end{pmatrix}\in\mathcal{A}(U)$ iff $T\in\mathcal{A}(S)$. As for the second assertion, let $A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\in\mathcal{B}(H^2\oplus H^2_-)$; then

$$AU-UA=\begin{pmatrix} A_{11}S-SA_{11}-K_1A_{21} & *\\ * & *\end{pmatrix}.$$

Hence $A_{11}=P_+A\restriction H^2\in\mathcal{A}(S)$ if $A\in\mathcal{A}(U)$, q.e.d. $\square$

Thus, $\mathcal{A}(S)$ is "smaller" then $\mathcal{A}(U)$ in the sense that the former is C^*-isomorphic to a subalgebra of the latter. All the same, it is still possible that $\mathcal{A}(S)$ and $\mathcal{A}(U)$ are actually C^*-isomorphic. We are only able to prove that, if such is the case, the isomorphism can not be spatial, i.e. of the form $T \mapsto WTW^*$, where $W : H^2 \to L^2(\mathbf{T})$ is unitary ($T \in \mathcal{A}(S)$, $WTW^* \in \mathcal{A}(U)$). Before exhibiting the proof, let us first establish some general properties of the algebras $\mathcal{A}$.

In general, there are two candidates for the definition of $\mathcal{A}$, which have happened to coincide in all cases encountered so far. Let H be a separable infinite-dimensional Hilbert space and $M \in \mathcal{B}(H)$. Define

$$\mathcal{A}(M) := \{T \in \mathcal{B}(H) : MT - TM \in \texttt{Comp}(H)\}$$

and

$$\mathcal{A}^\sharp(M) := \{T \in \mathcal{B}(H) : T - M^*TM \in \texttt{Comp}(H)\}.$$

The first investigation of $\mathcal{A}^\sharp(S)$ is reported to have been done by Barría and Halmos, whose results, unfortunately, appeared only as a rather unavailable preprint [4], and so remain unknown to the present author. Afterwards these spaces seem to have received almost no attention at all, although many results on essential commutants may be phrased in terms of them. Let us mention the theorem of Johnson and Parrot [20] which says that

$$\bigcap_{\phi \in L^\infty(\mathbf{T})} \mathcal{A}(M_\phi) = \{M_\phi + K : \phi \in L^\infty(\mathbf{T}),\ K \in \texttt{Comp}\},$$

where M_ϕ is the operator of multiplication by ϕ on $L^2(\mathbf{T})$, and two results of Davidson [11] concerning Toeplitz operators on H^2:

$$\bigcap_{\theta \text{ inner}} \mathcal{A}^\sharp(T_\theta) = \{T_\phi + K : \phi \in L^\infty(\mathbf{T}),\ K \text{ compact }\},$$

and

$$\bigcap_{\theta \text{ inner}} (\mathcal{A}(T_\theta) \cap \mathcal{A}(T_\theta^*)) = \{T_\phi + K : \phi \in QC,\ K \text{ compact}\}.$$

($QC = (H^\infty + C) \cap \overline{H^\infty + C}$ are the quasicontinuous functions on $\mathbf{T}$.)

The following proposition describes elementary properties of $\mathcal{A}(M)$ and $\mathcal{A}^\sharp(M)$.

Proposition 3.30. *(a) $\mathcal{A}(M)$ and $\mathcal{A}^\sharp(M)$ are norm-closed subspaces of $\mathcal{B}(H)$.*

(b) $\mathcal{A}(M)$ is an operator algebra with identity. It contains M, `Comp`*, and may be even all of $\mathcal{B}(H)$. It is a C^*-algebra if M is selfadjoint; in general, $\mathcal{A}(M^*) = \mathcal{A}(M)^*$.*

(c) $\mathcal{A}^\sharp(M)$ is a selfadjoint set containing `Comp`*. It may be all of $\mathcal{B}(H)$, but may also reduce merely to* `Comp`*.*

(d) $\mathcal{A}^\sharp(M)$ is C^*-algebra if $I - MM^* \in$ `Comp`*, and contains the identity if and only if $I - M^*M \in$* `Comp`*.*

(e) $\mathcal{A}(M) = \mathcal{A}^\sharp(M) \iff M$ is essentially unitary, and then it is a C^*-algebra with identity.

Proof. (a) Obvious.

(b) If $A, B \in \mathcal{A}(M)$, then $[AB, M] = A[B, M] + [A, M]B$ is also compact, hence $AB \in \mathcal{A}(M)$. The assertions $I, M \in \mathcal{A}(M)$, $\texttt{Comp} \subset \mathcal{A}(M)$ are immediate, and so is $\mathcal{A}(M^*) = \mathcal{A}(M)^*$. If M is selfadjoint, $M^* = M$, whence $\mathcal{A}(M) = \mathcal{A}(M)^*$ is also selfadjoint, and so is a C^*-algebra. If $M \in \texttt{Comp}$, $\mathcal{A}(M) = \mathcal{B}(H)$.

(c) The first two assertions are straightforward. If $M \in \texttt{Comp}$, $\mathcal{A}^\sharp(M) = \texttt{Comp}$; if $M = I$, $\mathcal{A}^\sharp(M) = \mathcal{B}(H)$.

(d) The second statement is clear; if $I - MM^* \in \texttt{Comp}$ and $A, B \in \mathcal{A}^\sharp(M)$, then

$$AB - M^*ABM = A(B - M^*BM) + (A - M^*AM)M^*BM - M^*A(I - MM^*)BM \in \texttt{Comp},$$

and so $AB \in \mathcal{A}^\sharp(M)$. Owing to (a) and (b), $\mathcal{A}^\sharp(M)$ must be a *-subalgebra of $\mathcal{B}(H)$, hence a C^*-algebra.

(e) If $\mathcal{A}(M) = \mathcal{A}^\sharp(M)$, then $I \in \mathcal{A}^\sharp(M)$ owing to (b), and so $I - M^*M \in \texttt{Comp}$; moreover, $M \in \mathcal{A}(M) \Rightarrow M \in \mathcal{A}^\sharp(M) \Rightarrow M^* \in \mathcal{A}^\sharp(M) \Rightarrow M^* \in \mathcal{A}(M)$ owing to (c), i.e. M is essentially normal. It follows that M is essentially unitary. On the other hand, if M is essentially unitary, then

$$A - M^*AM \in \texttt{Comp} \implies M(A - M^*AM) = MA - AM + (I - MM^*)AM \in \texttt{Comp} \implies$$

$$\implies MA - AM \in \texttt{Comp},$$

and

$$MA - AM \in \texttt{Comp} \implies M^*(MA - AM) = A - M^*AM - (I - M^*M)A \in \texttt{Comp} \implies$$

$$\implies A - M^*AM \in \texttt{Comp},$$

hence $\mathcal{A}(M) = \mathcal{A}^\sharp(M)$; it is a C^*-algebra by virtue of (d). $\square$

The last two properties suggest it might be helpful to try to describe $\mathcal{A}(M)$ and $\mathcal{A}^\sharp(M)$ in terms of the image m of M in the Calkin algebra

$$\texttt{Calk}(H) := \mathcal{B}(H)/\texttt{Comp}(H).$$

We will denote the images of $R, S, T, M, \dots \in \mathcal{B}(H)$ in $\texttt{Calk}(H)$ by corresponding small letters $r, s, t, m, \dots$, and by π the canonical projection $\mathcal{B}(H) \to \texttt{Calk}(H)$; π is a C^*-homomorphism.

Proposition 3.31. *(a) $T \in \mathcal{A}(M) \iff t \in \pi(\mathcal{A}(M))$, and similarly for $\mathcal{A}^\sharp(M)$.*
(b) $T \in \mathcal{A}(M) \iff t \in m'$, the commutant of m in $\texttt{Calk}(H)$.
(c) $T \in \mathcal{A}^\sharp(M) \iff t = m^*tm$.
(d) $\mathcal{A}(M)$ is a C^*-algebra iff m is a normal element of $\texttt{Calk}(H)$.
(e) $m'' \subset m'$; in particular, continuous functions of m belong to $\pi(\mathcal{A}(M))$.
(f) $\mathcal{A}(M) \subset \mathcal{A}(R)$ iff $r \in m''$.

Proof. (a) A direct consequence of the fact that $\mathcal{A}(M)$, resp. $\mathcal{A}^\sharp(M)$ contain $\texttt{Comp}$.

(b) and (c) Straightforward.

(d) $\mathcal{A}(M)$ is a C^*-algebra iff m' is. In that case, $m^* \in m'$, and so m is normal. On the other hand, if m is normal and $tm = mt$, then $t^*m = mt^*$ by the Putnam-Fuglede theorem; hence $t^* \in m'$, and so m' is a C^*-algebra.

(e) $m \in m'$ implies $m'' \subset m'$; continuous functions of m belong to m''; $\pi(\mathcal{A}(M)) = m'$ by (a).

(f) $\mathcal{A}(M) \subset \mathcal{A}(R)$ iff $m' \subset r'$. If this is the case, then $m'' \supset r''$ and so $r \in r'' \subset m''$. On the other hand, if $r \in m''$, then $r' \supset m''' = m'$. □

As a parting shot, we prove the promised theorem that the algebras $\mathcal{A}(U)$ and $\mathcal{A}(S)$ are not spatially isomorphic. A result of Johnson and Parrot, already alluded to above, will be needed:

Theorem 3.32. *If T is an operator on $L^2(\mathbf{T})$ such that $[T, M]$ is compact for all operators M of the form*

$$M = M_\phi + K, \qquad \phi \in L^\infty(\mathbf{T}), \quad K \text{ compact}, \tag{25}$$

then T is itself of this form.

Proof. See [20]. □

The Johnson-Parrot theorem may be rephrased in the following way: *Let*

$$\mathcal{L}_\infty := \{\pi(M_\phi) : \phi \in L^\infty(\mathbf{T})\} \subset \mathtt{Calk}(L^2(\mathbf{T})).$$

Then $\mathcal{L}_\infty$ coincides with its commutant: ${\mathcal{L}_\infty}' = \mathcal{L}_\infty$.

Lemma 3.33. *If an operator M of the form (25) is Fredholm, then* $\operatorname{ind} F = 0$.

Proof. We shall use the previous theorem[10]. If $M = M_\phi + K_1$ is Fredholm, then $m \equiv \pi(M) \in \mathtt{Calk}$ is invertible. If $t \in \mathcal{L}_\infty$, then $mt = tm$ and so

$$0 = m^{-1}(mt - tm)m^{-1} = tm^{-1} - m^{-1}t,$$

whence $m^{-1} \in {\mathcal{L}_\infty}'$. By virtue of the Johnson-Parrot theorem, $m^{-1} \in \mathcal{L}_\infty$; consequently, there exist $\psi \in L^\infty(\mathbf{T})$ and $K_2 \in \mathtt{Comp}$ such that

$$(M_\phi + K_1).M_\psi = I + K_2,$$

which implies that

$$M_{\phi\psi - 1} \in \mathtt{Comp}.$$

Let $\{z^n\}_{n \in \mathbf{Z}}$ be the standard basis of $L^2(\mathbf{T})$; since $z^n \xrightarrow{\mathrm{w}} 0$, the last condition forces

$$\|M_{\phi\psi-1} z^n\| \to 0.$$

But $\|M_{\phi\psi-1} z^n\|^2 = \|\phi\psi - 1\|_2^2$ for all $n \in \mathbf{Z}$; therefore $\phi\psi = \mathbf{1}$ and M_ψ is the inverse of M_ϕ. It follows that $\operatorname{ind} M_\phi = 0$, and so likewise $\operatorname{ind} M = \operatorname{ind}(M_\phi + K_1) = 0$. □

[10] A more elementary proof is bound to exist, but we won't bother with it, since we are going to use the Johnson-Parrot theorem in the sequel anyway.

Theorem 3.34. *Let S be the unilateral forward shift on H^2, U the bilateral forward shift on $L^2(\mathbf{T})$, $W : H^2 \to L^2(\mathbf{T})$ a unitary operator, and $Y = WSW^*$. Then it cannot happen that $\mathcal{A}(U) \subset \mathcal{A}(Y)$. In particular, the algebras $\mathcal{A}(U)$ and $\mathcal{A}(Y)$ are not spatially isomorphic.*

Proof. By virtue of the part (f) of the last proposition, $\mathcal{A}(U) \subset \mathcal{A}(Y)$ is equivalent to $y \in u''$. Since $U = M_z$ is of the form (25), we have $u \in \mathcal{L}_\infty$ and the Johnson-Parrot theorem implies that

$$u'' \subset \mathcal{L}_\infty{}'' = \mathcal{L}_\infty = \mathcal{L}_\infty{}' \subset u'.$$

Hence $y \in \mathcal{L}_\infty$, i.e. $Y = M_\phi + K$ for some $\phi \in L^\infty(\mathbf{T})$ and $K \in \texttt{Comp}$. Because $Y = WSW^*$,

$$\sigma_e(Y) = \sigma_e(S) = \mathbf{T},$$

so Y is Fredholm. In view of the preceding lemma, $\operatorname{ind} Y = 0$. But

$$\operatorname{ind} Y = \operatorname{ind}(WSW^*) = \operatorname{ind} S = -1$$

— a contradiction. Thus it can never happen that $\mathcal{A}(U) \subset \mathcal{A}(Y)$, or even $\mathcal{A}(U) = \mathcal{A}(Y)$. $\square$

We close this chapter with two open problems.

Problem. Are the C^*-algebras $\mathcal{A}(U)$ and $\mathcal{A}(S)$ (non-spatially) isomorphic?

Problem. Denote W_B and W_F the unitary operators

$$W_B : H^2 \to A^2(\mathbf{D}), \qquad z^n \in H^2 \mapsto \sqrt{n+1}\, z^n \in A^2(\mathbf{D}),$$

resp.

$$W_F : H^2 \to A^2(\mathbf{C}), \qquad z^n \in H^2 \mapsto (n!\, 2^n)^{-1/2} z^n \in A^2(\mathbf{C}),$$

constructed in the paragraph following Corollary 3.6 and in course of the proof of Theorem 3.24(iii), respectively. Denote further

$$B_\phi := W_B^* T_\phi W_B \qquad \text{for } \phi \in L^\infty(\mathbf{D}),$$

resp.

$$F_\phi := W_F^* T_\phi W_F \qquad \text{for } \phi \in L^\infty(\mathbf{C}),$$

and let $\mathcal{T}_B$, $\mathcal{T}_F$ be the C^*-subalgebras of $\mathcal{B}(H^2)$ generated by $\{B_\phi : \phi \in L^\infty(\mathbf{D})\}$ and $\{F_\phi : \phi \in L^\infty(\mathbf{C})\}$, respectively. Theorems 3.2 and 3.24 then assert that

$$\mathcal{T}_B \subset \mathcal{A}(S) \qquad \text{and} \qquad \mathcal{T}_F \subset \mathcal{A}(S).$$

Are these inclusions strict?

Chapter 4. THE TOEPLITZ CALCULUS

In this chapter, Toeplitz operators both on the Hardy space and the Bergman space, as well as on the Fock space, are dealt with; to prevent confusion, the following convention will be observed throughout: T_ϕ always denotes a Toeplitz operator on H^2 (i.e. $\phi \in L^\infty(\mathbf{T})$), B_ϕ stands for a Toeplitz operator on $A^2(\mathbf{D})$ ($\phi \in L^\infty(\mathbf{D})$), and F_ϕ for a Toeplitz operator on the Fock space $A^2(\mathbf{C})$ ($\phi \in L^\infty(\mathbf{C})$). The algebra $\mathcal{A}(S)$ will frequently be abbreviated to $\mathcal{A}$. When convenient, the spaces $A^2(\mathbf{D})$ and $A^2(\mathbf{C})$ will be identified with H^2 by means of the unitary operators W_B and W_F, respectively, mentioned at the very end of the previous chapter.

We begin by recalling several facts from the theory of Toeplitz operators on H^2. Denote

$$\tau_H := \{T_\phi : \phi \in L^\infty(\mathbf{T})\}$$

and let $\mathcal{T}_H$ be the C^*-algebra generated by τ_H. Further, define

(26)

$$\mathfrak{S}_0 := \{T \in \mathcal{B}(H^2) : \lim_{n\to\infty} \|Te_n\| = 0\} = \{T \in \mathcal{B}(H^2) : \lim_{n\to\infty} \|TS^n f\| = 0 \ \forall f \in H^2\},$$

where $S = T_z$ is the (forward) shift operator with respect to the standard basis $\{e_n\}_{n\in\mathbf{N}}$, $e_n(z) = z^n$, of H^2. The equality (26) is easily verified: if $Te_n \to 0$, then $TS^n p \to 0$ for every polynomial p; since these are dense in H^2 and the operators TS^n are uniformly bounded (by $\|T\|$), $TS^n f \to 0$ for any $f \in H^2$.

The following are classical results from the theory of Toeplitz operators on H^2. The abelianized algebra $\mathcal{T}_H$, i.e. $\mathcal{T}_H$ factored by its commutator ideal $\operatorname{Com}\mathcal{T}_H$, is (isometrically) C^*-isomorphic to $L^\infty(\mathbf{T})$, i.e. there is a contractive map

$$\xi : \mathcal{T}_H \to L^\infty(\mathbf{T}) \tag{27}$$

which is linear, multiplicative, surjective, preserves adjoints, and $\ker\xi = \operatorname{Com}\mathcal{T}_H$. The mapping

$$\phi \mapsto T_\phi$$

is an isometric cross-section of ξ, i.e. $\xi(T_\phi) = \phi$. Thus, every operator $T \in \mathcal{T}_H$ admits a unique decomposition

$$T = T_\phi + X, \qquad X \in \operatorname{Com}\mathcal{T}_H,\ \phi \in L^\infty(\mathbf{T}),$$

and $\phi = \xi(T)$; that's why the mapping ξ is sometimes called the *symbol map*. This map, in fact, can be described somewhat more explicitly: it can be shown that

$$\operatorname{Com}\mathcal{T}_H \subset \mathfrak{S}_0;$$

consequently, for arbitrary $T \in \mathcal{T}_H$, the limit

$$Mf := \lim_{n\to\infty} U^{*n} T S^n f$$

exists for every $f \in H^2$. Here H^2 is thought of as a subspace of $L^2(\mathbf{T})$ and U is the (bilateral, forward) shift operator on $L^2(\mathbf{T})$. The resulting operator $M : H^2 \to L^2(\mathbf{T})$

is bounded (by $\|T\|$, as a matter of fact) and commutes with U, so it must be of the form M_ϕ for some $\phi \in L^\infty(\mathbf{T})$. Now it's already easy to verify that $\phi = \xi(T)$. Thus

$$\xi(T) = \lim_{n\to\infty} U^{*n} T e_n \in L^\infty(\mathbf{T}) \subset L^2(\mathbf{T}).$$

We are not going to prove the facts mentioned above, but rather refer to the books of Douglas [12] or Nikolskiĭ[22].

Note also that, owing to the multiplicativity of ξ,

$$\xi(T_\phi T_\psi) = \xi(T_\phi)\xi(T_\psi) = \phi\psi = \xi(T_{\phi\psi}) \qquad \forall \phi, \psi \in L^\infty(\mathbf{T}), \tag{28}$$

which implies that $\ker \xi = \operatorname{Com} \mathcal{T}_H$ contains not only the commutator $[T_\phi, T_\psi]$, but even the *semi-commutator*

$$[T_\phi, T_\psi) := T_{\phi\psi} - T_\phi T_\psi$$

of T_ϕ and T_ψ.

The objective of this chapter is an attempt to construct a similar theory, or "symbol calculus", for the algebras $\mathcal{T}_B$, $\mathcal{T}_F$ or even $\mathcal{A}$. (The algebra $\mathcal{A}(U)$ shall not be explicitely considered, the reason being that the larger the algebra, the less chance to construct some calculus; and $\mathcal{A}(U)$ is "larger" than $\mathcal{A}(S) \equiv \mathcal{A}$ — cf. Theorem 3.29. All the same, most of the ideas below can be applied to $\mathcal{A}(U)$ as well.)

The most naive idea (e.g. for $\mathcal{T}_B$) is to try to obtain a continuous linear mapping

$$\xi_B : \mathcal{T}_B \to L^\infty(\mathbf{D}) \tag{29}$$

which is be multiplicative and admits $\phi \mapsto B_\phi$ as a cross-section. Unfortunately, this is doomed to a failure: the following lemma shows that $\operatorname{Com} \mathcal{T}_B \supset \mathtt{Comp}$, and so if ξ_B is to be multiplicative, its kernel must contain $\mathtt{Comp}$. But there exist non-zero compact Toeplitz operators on $A^2(\mathbf{D})$ (cf. Corollary 1.5), and so $\phi \mapsto B_\phi$ cannot be a cross-section of ξ_B. For $\mathcal{T}_F$ and $\mathcal{A}$ the same situation occurs.

Lemma 4.1. *(a) The algebras $\mathcal{T}_B$, $\mathcal{T}_F$ and $\mathcal{A}$ contain* $\mathtt{Comp}$.
(b) The commutator ideals Com $\mathcal{T}_B$, Com $\mathcal{T}_F$ and Com $\mathcal{A}$ contain $\mathtt{Comp}$.

Proof. (a) $\mathcal{T}_B$ and $\mathcal{T}_F$ contain $\mathtt{Comp}$ in view of Theorem 2.4, and $\mathcal{A} \supset \mathcal{T}_B$.

(b) The algebras $\mathcal{T}_B$, $\mathcal{T}_F$ and $\mathcal{A}$ are not commutative, so it suffices to show that $\mathtt{Comp}$ is the smallest proper (closed, two-sided, selfadjoint) ideal in each of them. We shall prove this for $\mathcal{T}_B$; the proof for $\mathcal{T}_F$ is similar, and $\operatorname{Com} \mathcal{A} \supset \operatorname{Com} \mathcal{T}_B$ since $\mathcal{A} \supset \mathcal{T}_B$.

So let $\mathcal{I}$ be a proper (closed, two-sided, $*-$) ideal in $\mathcal{T}_B$. Take a nonzero $T \in \mathcal{I}$ and vectors $x, y \neq 0$ such that $Tx = y$. In view of (a), the operators $A = \langle\,.\,, e\rangle x$ and $B = \langle\,.\,, y\rangle f$ belong to $\mathcal{T}_B$ for arbitrary e, f. Since $\mathcal{I}$ is an ideal, $BTA = \|y\|^2 \langle\,.\,, e\rangle f$ belongs to $\mathcal{I}$. It follows that $\mathcal{I}$ contains all rank one operators, hence all finite rank operators (by linearity), which are dense in $\mathtt{Comp}$. □

So let us give up multiplicativity, and ask only for a continuous linear map (29) which admits $\phi \mapsto B_\phi$ as a cross-section. Unfortunately, we are doomed to fail once again. The reason is that $\xi_B(B_\phi) = \phi$ implies $\|B_\phi\| \geq \|\xi_B\|^{-1} \|\phi\|_\infty$, i.e. the mapping

$\phi \mapsto B_\phi$ would have to be bounded below. This is, however, easily seen not to be the case. If χ is the characteristic function of the disc $\{z : |z|^2 \le R\}$, $0 < R < 1$, then B_ϕ is a diagonal operator with weights (cf. Proposition 1.3)

$$c_n = \int_0^R (n+1)t^n \, dt = \frac{R^{n+1}}{n+1},$$

which implies $\|B_\phi\| = R$ while $\|\phi\|_\infty = 1$.

This suggests that, perhaps, there is nothing wrong with multiplicativity, but rather with the target space $L^\infty(\mathbf{D})$; so let us insist on multiplicativity and try to replace $L^\infty(\mathbf{D})$ by something else. A natural candidate for the "something" turns up quickly. Namely, the C^*-algebras $\mathcal{A}/\mathrm{Com}\,\mathcal{A}$, $\mathcal{T}_B/\mathrm{Com}\,\mathcal{T}_B$ and $\mathcal{T}_F/\mathrm{Com}\,\mathcal{T}_F$ are commutative, and therefore C^*-isomorphic to the spaces of all continuous functions on their maximal ideal spaces, via the Gelfand transform. It remains to describe these spaces and see whether they are not somehow connected with $L^\infty(\mathbf{D})$ (or $L^\infty(\mathbf{C})$, respectively) — for instance, they might be homeomorphic to some subsets of the maximal ideal space of $L^\infty(\mathbf{D})$ (or of $L^\infty(\mathbf{C})$, for that matter — these two are homeomorphic) , or something like that. This description in turn amounts to identifying the multiplicative linear functionals on $\mathcal{A}/\mathrm{Com}\,\mathcal{A}$, $\mathcal{T}_B/\mathrm{Com}\,\mathcal{T}_B$ and $\mathcal{T}_F/\mathrm{Com}\,\mathcal{T}_F$, respectively — or, which is the same, on $\mathcal{A}$, $\mathcal{T}_B$, $\mathcal{T}_F$. Thus we are lead to a fundamental question: are there any multiplicative linear functionals on these algebras at all?

Before going on, let us make two remarks. First, one might try to obtain a multiplicative linear functional e.g. on $\mathcal{A}$ by restricting to $\mathcal{A}$ a multiplicative linear functional on $\mathcal{B}(H^2)$. This is, however, impossible — there are no multiplicative linear functionals on $\mathcal{B}(H^2)$ and, consequently, $\mathrm{Com}\,\mathcal{B}(H^2) = \mathcal{B}(H^2)$. To see this, decompose H^2 into $H_1 \oplus H_2$, where both H_1 and H_2 are infinite-dimensional. Then there exists a unitary operator which maps H_1 onto H_2 and vice versa; denote it V, and let P_i stand for the orthogonal projection onto H_i $(i = 1, 2)$. Now, if ϕ were a nontrivial multiplicative linear functional on $\mathcal{B}(H^2)$,

$$P_1 + P_2 = I \implies \phi(P_1) + \phi(P_2) = 1,$$

$$V^*V = I \implies \overline{\phi(V)}\phi(V) = 1,$$

$$V^*P_1V = P_2 \implies \phi(P_2) = \phi(V^*)\phi(P_1)\phi(V) = \phi(P_1),$$

so $\phi(P_1) = \phi(P_2) = 1/2$; but

$$P_1P_2 = 0 \implies \frac{1}{4} = \phi(P_1)\phi(P_2) = 0,$$

a contradiction.

Second, we have noted that the symbol map ξ for Toeplitz operators on H^2 satisfies (28)

$$\xi(T_\phi T_\psi - T_{\phi\psi}) = 0.$$

It should be pointed out that this relation cannot be satisfied on the Bergman space $A^2(\mathbf{D})$ or the Fock space $A^2(\mathbf{C})$, i.e. for algebras $\mathcal{T}_B$ or $\mathcal{T}_F$ (or $\mathcal{A}$): the following example shows that there even exists $\phi \in L^\infty(\mathbf{D})$ such that $[B_\phi^*, B_\phi)$ is a nonzero multiple of the identity.

Example 4.2. Define the function $\phi \in L^\infty(\mathbf{D})$ by

$$\phi(z) = \exp(i \ln\ln \frac{1}{|z|^2}).$$

In view of Proposition 1.3, B_ϕ is a diagonal operator with weights

$$c_n = \int_0^1 \exp(i \ln\ln \frac{1}{t}) \,.\, (n+1)t^n \, dt.$$

Changing the variable to $w = z^{n+1}$ gives

$$c_n = \int_0^1 \exp(i \ln\ln \frac{1}{w} - i\ln(n+1)) \, dw = e^{-i\ln(n+1)} \int_0^1 \exp(i \ln\ln \frac{1}{w}) \, dw.$$

Substituting once more, namely, $y = \ln \dfrac{1}{w}$, yields

$$\int_0^1 \exp(i \ln\ln \frac{1}{w}) \, dw = \int_0^{+\infty} \exp(i \ln y) \,.\, e^{-y} \, dy = \Gamma(i+1) \qquad \text{(popularly "}i!\text{")}.$$

Also, $B_{\overline{\phi}} = B_\phi^* = \operatorname{diag}(\overline{c_n})$ and $B_{\overline{\phi}\phi} = I$ since ϕ is unimodular on $\mathbf{D}$. Summing up, we have

$$B_{\overline{\phi}\phi} - B_{\overline{\phi}} B_\phi = \operatorname{diag}(1 - |c_n|^2) = \operatorname{diag}(1 - |\Gamma(i+1)|^2) = (1 - |\Gamma(i+1)|^2).I$$

It remains to show that $|\Gamma(i+1)| \neq 1$. To that aim, recall the formulas for the gamma function

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(\overline{x}) = \overline{\Gamma(x)}, \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

and compute:

$$|\Gamma(i+1)|^2 = \Gamma(i+1)\Gamma(1-i) = i\Gamma(i)\Gamma(1-i) =$$
$$= \frac{\pi i}{\sin \pi i} = \frac{2\pi}{e^\pi - e^{-\pi}} = 0,272\cdots < 1.$$

On the Fock space, a similar counterexample can be constructed. The function $\phi \in L^\infty(\mathbf{C})$,

$$\phi(z) = \exp(i|z|^2/2),$$

is unimodular; the corresponding Toeplitz operator F_ϕ is diagonal (cf. Proposition 1.7) with weights

$$c_n = \int_0^{+\infty} e^{it} \,.\, \frac{t^n e^{-t}}{n!} \, dt.$$

Expanding e^{it} into the Taylor series and integrating term by term[11] yields

$$c_n = \sum_{k=0}^{\infty} \binom{n+k}{k} i^k = (1-i)^{-n-1}.$$

Consequently, $F_{\overline{\phi}\phi} - F_{\overline{\phi}} F_\phi = \operatorname{diag}(1-|c_n|^2) = \operatorname{diag}(1-2^{-n-1})$, which is an invertible operator and so cannot belong to $\ker \xi_F$ for any nonzero ξ_F. $\square$

Now let us come back to the search of multiplicative linear functionals on the three algebras $\mathcal{T}_B$, $\mathcal{T}_F$ and $\mathcal{A}$. Owing to Lemma 4.1, every multiplicative linear functional on $\mathcal{A}$ must vanish on $\mathtt{Comp}$; consequently, it suffices to look for multiplicative linear functionals on $\mathcal{A}/\mathtt{Comp}$, and similarly for $\mathcal{T}_B$ and $\mathcal{T}_F$. These are subalgebras of the Calkin algebra $\mathtt{Calk}(H^2) = \mathcal{B}(H^2)/\mathtt{Comp}(H^2)$, which suggests that some techniques used for the study of $\mathtt{Calk}$ might turn useful. The algebra $\mathcal{A}/\mathtt{Comp} \subset \mathtt{Calk}$ admits a particularly easy description (cf. Proposition 3.31(b)): it coincides with the commutant of the image $s = \pi(S)$ of the shift operator S in $\mathtt{Calk}(H^2)$. For this reason, we shall be concerned mainly with $\mathcal{A}$ from now on, but most of what is said applies to $\mathcal{T}_B$ and $\mathcal{T}_F$ as well.

In the classical (i.e. H^2) case, a powerful tool is provided by the dilation theory, which enables one to "lift", in some sense, a Toeplitz operator T_ϕ from H^2 to an operator on a larger space (namely, $L^2(\mathbf{T})$) enjoying certain properties (see e.g. [25]). It turns out that a similar lifting can be constructed using certain representation of the Calkin algebra. The classical lifting mentioned above (i.e. of T_ϕ to $L^2(\mathbf{T})$) may then be obtained by passing to certain subspace, isomorphic to $L^2(\mathbf{T})$ (the subspace is the same for all operators T_ϕ). We proceed to describe the representation of $\mathtt{Calk}(H^2)$; the idea goes back to Calkin [10].

Denote by π the canonical projection of $\mathcal{B}(H^2)$ onto $\mathtt{Calk}(H^2)$; to simplify the notation, we shall often write (once again) t, s, etc. for $\pi(T)$, $\pi(S)$, etc.

We begin by a brief excursion into Banach limits. Loosely speaking, they are extensions of the usual "lim" to the space l^∞ of all bounded sequences of complex numbers. In more precise terms, a Banach limit is a mapping Lim from l^∞ into $\mathbf{C}$ which satisfies the following conditions:

(C1) it is linear, of norm 1, and extends the usual lim (i.e. $\operatorname{Lim} f_n = \lim f_n$ whenever the latter exists);

(C2) it preserves complex conjugation, i.e. $\operatorname{Lim} \overline{f_n} = \overline{\operatorname{Lim} f_n} \quad \forall \{f_n\} \in l^\infty$); consequently, $\operatorname{Lim} f_n$ is a real number when f_n is real for all $n \in \mathbf{N}$;

(C3) it is positive, i.e. $f_n \geq 0 \ \forall n \in \mathbf{N}$ implies $\operatorname{Lim} f_n \geq 0$.

To obtain such a functional, one may procced as follows. First, use the Hahn-Banach theorem to extend "lim" to a linear functional on l^∞ without increasing the norm; this gives a functional "Lim" which satisfies (C1). Next, replace Lim by $\operatorname{Lim} \dfrac{f_n + \overline{f_n}}{2}$ if necessary; the resulting functional satisfies both (C1) and (C2). Finally, observe that (C1) and (C2) already imply (C3). It suffices to show that if $0 \leq f_n \leq 1$ for all

[11] This is legal for $\int_0^1$, since the Taylor series converges uniformly for $0 < t < 1$; as for $\int_1^{+\infty}$, split the series into four (corresponding to $i^k = \pm 1, \pm i$) and apply the Lebesgue Monotone Convergencs Theorem to each.

n, then $0 \leq \operatorname{Lim} f_n \leq 1$. But

$$(0 \leq f_n \leq 1 \ \forall n) \iff (f_n \text{ is real, } |f_n| \leq 1 \text{ and } |1 - f_n| \leq 1 \ \forall n) \implies$$

$$\implies (\operatorname{Lim} f_n \text{ is real, } |\operatorname{Lim} f_n| \leq 1 \text{ and } |1 - \operatorname{Lim} f_n| \leq 1) \iff$$

$$(\text{by (C1) and (C2)}) \qquad \iff 0 \leq \operatorname{Lim} f_n \leq 1.$$

Thus, Banach limits exist in abundance. There are two additional conditions one could impose:

(C4) $\operatorname{Lim} f_{n+1} = \operatorname{Lim} f_n$;

(C5) $\operatorname{Lim} f_n g_n = \operatorname{Lim} f_n \operatorname{Lim} g_n$.

An unpleasant thing is that these two conditions conflict with each other: let

$$x_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}, \qquad y_n = x_{n+1}.$$

If there were a "Lim" which satisfies both (C4) and (C5), we would have $\operatorname{Lim} x_n = \operatorname{Lim} y_n$ (by (C4)) and $\operatorname{Lim}(x_n + y_n) = 1$, whence $\operatorname{Lim} x_n = \operatorname{Lim} y_n = 1/2$; but $\operatorname{Lim} x_n \,.\, \operatorname{Lim} y_n = 0$ by (C5) — a contradiction.

All the same, it is quite easy to construct a Banach limit which satisfies either only (C4), or only (C5). The former may be obtained from every Banach limit by replacing it with $\operatorname{Lim}\left(\frac{1}{n+1}\sum_{k=0}^{n} x_k\right)$. This does not affect (C1) — (C3) and

$$\operatorname{Lim}\left(\frac{1}{n+1}\sum_{k=0}^{n} x_k\right) - \operatorname{Lim}\left(\frac{1}{n+1}\sum_{k=0}^{n} x_{k+1}\right) = \operatorname{Lim}\frac{x_0 - x_{n+1}}{n+1} = 0.$$

To obtain a Banach limit satisfying (C5), adopt a different approach. With pointwise multiplication, l^∞ is a commutative Banach algebra, and hence is isomorphic, by means of the Gelfand transform, to the space $C(\mathfrak{M})$ of all continuous (complex-valued) functions on its maximal ideal space $\mathfrak{M}$. This space is usually denoted $\beta\mathbf{N}$, since it coincides with the Stone-Čech compactification of the set $\mathbf{N}$ equipped with the discrete topology. The elements of $\mathfrak{M} = \beta\mathbf{N}$ are multiplicative linear functionals on $\mathbf{N}$ and are known to satisfy (C1) — (C3) and (C5), except for the fact that they need not extend the functional "lim". The set $\mathbf{N}$ can be embedded into $\beta\mathbf{N}$ in a natural way; the multiplicative linear functional corresponding to $n \in \mathbf{N}$ is given by

$$\widehat{n} : \{x_k\} \in l^\infty \mapsto x_n \in \mathbf{C}.$$

We assert that if $\phi \in \beta\mathbf{N} \setminus \mathbf{N}$, then ϕ extends "lim". To prove this, it suffices to check that $\phi(\{x_n\}) = 0$ whenever $\lim_{n\to\infty} x_n = 0$. Since $\mathbf{N}$ is dense in $\beta\mathbf{N}$, there exists a net $\{n_\iota\}_{\iota\in\Lambda} \subset \mathbf{N}$ such that $\widehat{n}_\iota \to \phi$. If there were a number $m \in \mathbf{N}$ which occured in $\{n_\iota\}_{\iota\in\Lambda}$ an infinite number of times, we would have $\phi = m$, contrary to our assumption $\phi \in \beta\mathbf{N} \setminus \mathbf{N}$. It follows that $n_\iota \to \infty$, whence

$$\phi(\{x_k\}) = \lim_{\iota\in\Lambda} \widehat{n}_\iota(\{x_k\}) = \lim_{\iota\in\Lambda} x_{n_\iota} = 0$$

whenever $\lim_{k\to\infty} x_k = 0$, and our claim is verified.

Thus, Banach limits satisfying either (C4) or (C5) exist in abundance, too. In the sequel, a Banach limit, always denoted "Lim", is assumed to satisfy only (C1) — (C3); it will be pointed out explicitely when one of the additional properties is required.

Now we may proceed to introduce the Calkin representation. Let

$$\mathcal{L}'' := \Big\{ \{x_k\}_{k\in\mathbf{N}} : x_k \in H^2 \ \forall k \in \mathbf{N}, \text{ and } x_k \xrightarrow{\mathrm{w}} 0 \Big\}$$

be the set of all sequences of elements of H^2 which tend weakly to zero. With componentwise addition and scalar multiplication, $\mathcal{L}''$ becomes a vector space. Let Lim be a Banach limit. The functional[12]

$$\|x\|_{\mathcal{L}} := (\mathrm{Lim}\, \|x_k\|_{H^2}^2)^{1/2}, \qquad x = \{x_k\} \in \mathcal{L}''$$

is easily seen to be a pseudonorm on $\mathcal{L}''$. The factor space

$$\mathcal{L}' := \mathcal{L}''/\{x \in \mathcal{L}'' : \|x\|_{\mathcal{L}} = 0\}$$

is a normed linear space with respect to the norm $\|x\|_{\mathcal{L}}$; in fact, it is even a pre-Hilbert space with respect to the scalar product

$$\langle x, y\rangle_{\mathcal{L}} := \mathrm{Lim}\langle x_n, y_n\rangle, \qquad x, y \in \mathcal{L}''.$$

Let $\mathcal{L}$ be the completion of $\mathcal{L}'$. Then $\mathcal{L}$ is a Hilbert space and $\mathcal{L}'$ is a dense subset of $\mathcal{L}$. For notational convenience, the elements of $\mathcal{L}'$ will be denoted simply as $x = \{x_n\}$, $y = \{y_k\}$, etc., which is not so cumbersome as writing rigorously $x \bmod \{w \in \mathcal{L}'' : \|w\|_{\mathcal{L}} = 0\}$, or something similar.

Remark 4.3. The space $\mathcal{L}$ is uncomfortably large — it is not separable. This can be seen as follows. Let $\{q_n\}_{n\in\mathbf{N}}$ be the set of all rational numbers ($q_m \neq q_k$ if $m \neq k$). For each irrational number α, select a sequence $\Lambda_\alpha \subset \mathbf{N}$ such that $\lim_{j\in\Lambda_\alpha} q_j = \alpha$. Thus if $\alpha \neq \beta$ are irrational numbers, the sequences Λ_α and Λ_β have at most a finite number of terms in common. Let X_α be the sequence $\{e_j\}_{j\in\Lambda_\alpha}$, where $\{e_j\}_{j\in\mathbf{N}}$ is, as before, the standard orthonormal basis of H^2. Then $X_\alpha \in \mathcal{L}''$ and

$$\|X_\alpha - X_\beta\|_{\mathcal{L}}^2 = \mathrm{Lim}\, \|e_{\Lambda_\alpha(j)} - e_{\Lambda_\beta(j)}\|^2 = 2$$

for arbitrary irrational numbers $\alpha \neq \beta$. Since there are uncountably many irrational numbers, $\mathcal{L}$ must be inseparable. $\square$

Let T be a bounded linear operator on H^2. If $x_n \xrightarrow{\mathrm{w}} 0$ in H^2, then also $Tx_n \xrightarrow{\mathrm{w}} 0$. Consequently, the mapping

$$\{x_k\} \mapsto \{Tx_k\}$$

maps $\mathcal{L}''$ into $\mathcal{L}''$; because

$$\mathrm{Lim}\, \|Tx_k\|^2 \leq \mathrm{Lim}(\|T\|^2 \,.\, \|x_k\|^2) = \|T\|^2 \,.\, \mathrm{Lim}\, \|x_k\|^2,$$

[12] If Lim satisfies (C5), $\|x\|_{\mathcal{L}}$ equals $\mathrm{Lim}\, \|x_k\|$.

it does not increase norms, and so can be extended to a bounded linear operator on $\mathcal{L}$, which will be denoted $T^\sharp$. Thus $T^\sharp$ is a linear operator on $\mathcal{L}$ and

$$\|T^\sharp\|_{\mathcal{L}\to\mathcal{L}} \le \|T\|.$$

Let us establish some properties of the space $\mathcal{L}$ and the transformation $T \mapsto T^\sharp$ which will be needed in the sequel.

Proposition 4.4. *The transformation $T \mapsto T^\sharp$ is linear and contractive, $(T^*)^\sharp = (T^\sharp)^*$ and $(T_1T_2)^\sharp = T_1^\sharp T_2^\sharp$.*

Proof. The first two assertions have been verified in the preceding paragraph; as for the other two, it suffices to check them on elements of $\mathcal{L}''$, which reduces to direct consequences of the properties of Banach limits. □

Proposition 4.5. *$T^\sharp = 0$ iff $T \in$ `Comp`.*

Proof. The last proposition implies that the kernel of the transformation $T \mapsto T^\sharp$ is an ideal (two-sided, closed, selfadjoint) in $\mathcal{B}(H^2)$. There are only three such ideals : $\{0\}$, `Comp` and the whole $\mathcal{B}(H^2)$. Since $(I_{H^2})^\sharp = I_{\mathcal{L}}$, it suffices to show that $K^\sharp = 0$ if $K \in$ `Comp`. But $K \in$ `Comp` and $x_n \xrightarrow{\text{w}} 0$ implies $\|Kx_n\| \to 0$, i.e. $K^\sharp x = 0$ for $x \in \mathcal{L}''$; by continuity, $K^\sharp = 0$. □

Corollary 4.6. *$\|T^\sharp\| \le \|T\|_e$, the essential norm of T. Consequently, the transformation $T \mapsto T^\sharp$ induces a mapping $t \mapsto t^\sharp$ from the Calkin algebra `Calk`(H^2) into $\mathcal{B}(\mathcal{L})$. This mapping is an (isometric) C^*-isomorphism of `Calk` onto a C^*-subalgebra of $\mathcal{B}(\mathcal{L})$. Hence, we have even $\|T^\sharp\| = \|T\|_e$.*

Proof. The preceding two propositions imply that

$$\|T^\sharp\| = \|T^\sharp + K^\sharp\| \le \|T + K\|$$

for every compact operator K; hence $\|T^\sharp\| \le \|T\|_e$. The rest follows from Propositions 4.4 & 4.5 and the fact that an injective C^*-homomorphism is isometric. □

Corollary 4.7. *$T \in \mathcal{A} \iff t \in s' \iff T^\sharp \in S^{\sharp\prime}$.*

Proof. Immediate from the last corollary. □

Proposition 4.8. *(1) The mapping*

$$J : x \in H^2 \mapsto \{S^n x\}_{n\in\mathbf{N}} \in \mathcal{L}''$$

is an isometrical isomorphism of H^2 onto a subspace $\mathcal{H} \subset \mathcal{L}$. An orthonormal basis for $\mathcal{H}$ is given by

$$E_n = Je_n = S^{\sharp n} E_0 = \{e_{n+k}\}_{k\in\mathbf{N}} \in \mathcal{L}'', \qquad n \in \mathbf{N}, \tag{30}$$

where $\{e_k\}_{k\in\mathbf{N}}$ is the standard orthonormal basis of H^2.

(2) The operator $S^\sharp$ behaves on $\mathcal{H}$ as S on H^2: $\mathcal{H}$ is an invariant subspace for $S^\sharp$ and $S^\sharp J = JS$, i.e. $S^\sharp E_N = E_{N+1}$.

(3) The operator $S^\sharp \in \mathcal{B}(\mathcal{L})$ is unitary.

Proof. (1) Since $S^{*k} \to 0$ in SOT, $S^k x \xrightarrow{\text{w}} 0$ for every $x \in H^2$, so $\{S^k x\}_{k\in\mathbf{N}} \equiv Jx \in \mathcal{L}''$. Further,

$$\|Jx\|^2_{\mathcal{L}} = \operatorname{Lim} \|S^k x\|^2 = \operatorname{Lim} \|x\|^2 = \|x\|^2,$$

i.e. J is an isometry. Since $\{e_n\}$ is an orthonormal basis for H^2, $\{Je_n\}$ must be an orthonormal basis for $\mathcal{H} = \operatorname{Ran} J$; the formulas (30) are immediate.

(2) It suffices to check that $S^\sharp E_N = E_{N+1}$, and that's immediate from the definitions.

(3) $S^*S = I$, hence $S^{\sharp *}S^\sharp = I_{\mathcal{L}}$; $I - SS^*$ is a compact operator (namely, $\langle\,.\,, e_0\rangle e_0$), hence $I_{\mathcal{L}} - S^\sharp S^{\sharp *} = 0$. So, indeed, $S^\sharp$ is unitary. $\square$

The last Proposition has an interesting corollary. The shift operator S on H^2 has a minimal unitary dilation (in fact, a minimal unitary extension) sensu Nagy-Foias; the latter, in fact, can be identified with the bilateral shift operator $U = M_z$ on $L^2(\mathbf{T})$. Owing to parts (2) and (3) of the last Proposition, $S^\sharp$ is a unitary extension (hence, of course, also a unitary dilation) of the operator $S^\sharp\restriction\mathcal{H}$ which is unitarily equivalent to the operator S on H^2. It follows that $S^\sharp$ must contain the minimal unitary dilation $U = M_z$ of S — there has to be a subspace $\mathcal{K}$ of $\mathcal{L}$ which contains $\mathcal{H}$, is invariant under $S^\sharp$, and the restriction $S^\sharp\restriction\mathcal{K}$ is unitarily equivalent to M_z on $L^2(\mathbf{T})$. This subspace can be, in fact, described explicitly: it is $\mathcal{K} = \bigvee_{n\in\mathbf{Z}} S^{\sharp n}E_0$. The vectors $E_N := S^{\sharp N}E_0$, $N \in \mathbf{Z}$, form an orthonormal basis for $\mathcal{K}$. These vectors belong to $\mathcal{L}''$, and so may be written down explicitely:

$$E_N = \begin{cases} \{\underbrace{0,\ldots,0}_{-N \text{ zeroes}}, e_0, e_1, e_2, \ldots\} & \text{if } N < 0, \\ \{e_N, e_{N+1}, e_{N+2}, \ldots\} & \text{if } N \geq 0 \qquad \text{(as in (30))}. \end{cases}$$

Both expressions may be written as $E_N = \{S^{k+N}e_0\}_{k\in\mathbf{N}}$ if we agree to let the undefined terms (for $N < 0$, $0 \leq k < -N$) be zero. The isometry $J : H^2 \to \mathcal{H}$ may be extended to a unitary map of $L^2(\mathbf{T})$ onto $\mathcal{K}$ by setting

$$J(\overline{z}^N) = E_{-N} \qquad \text{for } N \geq 0.$$

Clearly the subspace $\mathcal{K} \subset \mathcal{L}$ reduces $S^\sharp$.

Remark 4.9. The orthogonal projection P of $\mathcal{L}$ onto $\mathcal{H}$ also admits an explicit description. Since J is an isometry of H^2 onto $\mathcal{H}$, P is the orthogonal projection onto $\operatorname{Ran} J$, so $P = JJ^*$. Consequently, for $\{x_n\} \in \mathcal{L}''$, $Px = Jh$ where $h = J^*x \in H^2$ is characterized by

$$\langle h, g\rangle_{H^2} = \langle J^*x, g\rangle_{H^2} = \langle x, Jg\rangle_{\mathcal{L}} = \operatorname{Lim}\langle x_n, S^n g\rangle = \operatorname{Lim}\langle S^{*n}x_n, g\rangle \qquad \forall g \in H^2,$$

i.e. h is the "weak limit with respect to Lim" of the sequence $S^{*n}x_n$. $\square$

Our next aim is to determine $T \in \mathcal{B}(H^2)$ for which $T^\sharp\mathcal{K} \subset \mathcal{K}$ and $T^\sharp\restriction\mathcal{K} = 0$, respectively.

Proposition 4.10. *Suppose that $T \in \mathcal{A}$ or that* Lim *satisfies (C4). Then $T^\sharp\mathcal{K} \subset \mathcal{K}$ if and only if*

$$\sum_{n\in\mathbf{Z}} |\operatorname*{Lim}_k t_{k,n+k}|^2 = \operatorname*{Lim}_k \sum_{n=0}^{\infty} |t_{k,n}|^2, \qquad \textit{where } t_{k,n} := \langle Te_k, e_n\rangle. \tag{31}$$

Visually, in the matrix of T with respect to the orthonormal basis $\{e_n\}$, the Lim's on the left-hand side are Limits of the entries on diagonals parallel to the main diagonal, whereas on the right-hand side is the Limit of norms of the columns of the matrix.

Proof. $T^\sharp\mathcal{K} \subset \mathcal{K}$ is equivalent to $T^\sharp E_N \in \mathcal{K}\ \forall N \in \mathbf{N}$.

For $T \in \mathcal{A}$, $T^\sharp S^\sharp = S^\sharp T^\sharp$, so $T^\sharp E_N = T^\sharp S^{\sharp N} E_0 = S^{\sharp N} E_0$; since $\mathcal{K}$ reduces $S^\sharp$, $T^\sharp E_N \in \mathcal{K} \iff T^\sharp E_0 \in \mathcal{K}$. Because $\{E_N\}$ is a basis of $\mathcal{K}$, this is in turn equivalent to

$$\|T^\sharp E_0\|^2 = \sum_{M\in\mathbf{N}} |\langle T^\sharp E_0, E_M\rangle|^2. \tag{32}$$

Now $\|T^\sharp E_0\|^2 = \operatorname{Lim}_k \|TS^k e_0\|^2$, which is the right-hand side of (31), while

$$\langle T^\sharp E_0, E_M\rangle = \operatorname*{Lim}_k \langle TS^k e_0, S^{M+k} e_0\rangle = \operatorname*{Lim}_k \langle Te_k, e_{M+k}\rangle,$$

and so (32) is equivalent to (31).

For a Lim satifying (C4), we have to check (32) not only for E_0, but for all E_N,; but

$$\|T^\sharp E_N\|^2 = \operatorname*{Lim}_k \|TS^{N+k} e_0\|^2 = \operatorname{Lim} \|TS^k e_0\|^2$$

$$\langle T^\sharp E_N, E_M\rangle = \operatorname{Lim}\langle TS^{N+k} e_0, S^{M+k} e_0\rangle = \operatorname{Lim}\langle Te_k, e_{M-N+k}\rangle,$$

the last equalities on each line being a consequence of (C4); thus we arrive at (31) once again. □

The last criterion is rather discouraging; not only because it is somewhat complicated to check, but also because it depends on the Banach limit chosen. To see this, restrict attention to diagonal operators. If $T = \operatorname{diag}(c_n)$, $c_n \in \mathbf{C}$, then (31) reduces to

$$|\operatorname{Lim} c_k|^2 = \operatorname{Lim} |c_k|^2.$$

This always holds if Lim satisfies (C5), i.e. is multiplicative; as we have seen, it cannot satisfy (C4) in that case, and so our criterion says exactly that $T^\sharp\mathcal{K} \subset \mathcal{K}$ for all diagonal operators T belonging to $\mathcal{A}$ (i.e. subject to $c_k - c_{k+1} \to 0$). On the other hand, if we take $c_k = (-1)^k$, then $\operatorname{Lim} c_k = 0$ for every Lim satisfying (C4), while $\operatorname{Lim} |c_k|^2 = 1$. All one can say for general Lim's is that $|\operatorname{Lim} c_k|^2 \le \operatorname{Lim} |c_k|^2$, the equality taking place iff $\operatorname{Lim} |c_k - c|^2 = 0$, where $c = \operatorname{Lim} c_k$. In fact,

$$\begin{aligned}\operatorname{Lim} |c_k - c|^2 &= \operatorname{Lim}(|c_k|^2 - c_k\overline{c} - c\overline{c_k} + |c|^2) = \\ &= \operatorname{Lim} |c_k|^2 - \operatorname{Lim} c_k \,.\, \overline{c} - c.\overline{\operatorname{Lim} c_k} + |c|^2 = \operatorname{Lim} |c_k|^2 - |c|^2.\end{aligned}$$

The criterion for $T^\sharp{\restriction}\mathcal{K} = 0$ is much simpler. Define

$$\widetilde{\mathfrak{S}}_0 := \{T \in \mathcal{B}(H^2) : \operatorname{Lim} \|Te_n\|^2 = 0\}.$$

If Lim satisfies (C4), $T \in \widetilde{\mathfrak{S}}_0$ implies that $\operatorname{Lim}\|TS^n f\|^2 = 0$ for all $f \in H^2$. This is obvious when f is a polynomial (i.e. a linear combination of e_n); for general $f \in H^2$, take a polynomial p such that $\|f-p\| < \epsilon$ and observe that $\operatorname{Lim}\|TS^n(f-p)\|^2 \le \|T\|^2\epsilon^2$, so $\operatorname{Lim}\|TS^n f\|^2 \le 2\|T\|^2\epsilon^2$; since $\epsilon > 0$ may be arbitrary, the claim follows.

Proposition 4.11. *Suppose that $T \in \mathcal{A}$ or that* Lim *satisfies (C4). Then $T^\sharp{\restriction}\mathcal{K} = 0$ if and only if $T \in \widetilde{\mathfrak{S}}_0$.*

Proof. $T^\sharp{\restriction}\mathcal{K} = 0$ if and only if $T^\sharp E_N = 0$ for all $N \in \mathbf{Z}$.

If $T \in \mathcal{A}$, $T^\sharp E_N = S^{\sharp N} T^\sharp E_0$, so it suffices to determine when $T^\sharp E_0 = 0$. But

$$\|T^\sharp E_0\|^2 = \operatorname{Lim}\|TS^n e_0\|^2 = \operatorname{Lim}\|Te_n\|^2,$$

and the assertion follows.

If Lim satisfies (C4), we have

$$\|T^\sharp E_N\|^2 = \operatorname*{Lim}_k \|TS^{k+N}e_0\|^2 = \operatorname{Lim}\|TS^k e_0\|^2,$$

which vanishes iff $T \in \widetilde{\mathfrak{S}}_0$. $\square$

Now let us see how the above apparatus applies to Toeplitz operators on the Hardy space H^2. For $T = T_\phi$, $\phi \in L^\infty(\mathbf{T}) \subset L^2(\mathbf{T})$, we have $\langle Te_k, e_n\rangle = \phi_{n-k}$ (the Fourier coefficient). Hence, the condition (31) reads

$$\sum_{n\in\mathbf{Z}} |\phi_n|^2 = \operatorname*{Lim}_k \sum_{n=-k}^{\infty} |\phi_n|^2,$$

which is certainly true; thus, $T_\phi^\sharp\mathcal{K} \subset \mathcal{K}$. Via the isomorphism J between $\mathcal{K}$ and $L^2(\mathbf{T})$, the operator $T_\phi^\sharp{\restriction}\mathcal{K}$ induces some operator on $L^2(\mathbf{T})$; let us identify it. One has[13]

$$\langle T_\phi^\sharp E_N, E_M\rangle = \operatorname*{Lim}_k \langle T_\phi S^{k+N}e_0, S^{k+M}e_0\rangle = \operatorname*{Lim}_k \int_{\mathbf{T}} \phi(z) z^{k+N}\overline{z}^{k+M}\,dz = \phi_{M-N},$$

and so $T_\phi^\sharp J = JM_\phi$ on $L^2(\mathbf{T})$, where $M_\phi : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ is the operator of multiplication by $\phi \in L^\infty(\mathbf{T})$.

Next, since $T_\phi^\sharp\mathcal{K} \subset \mathcal{K}$ for all $\phi \in L^\infty(\mathbf{T})$, we must also have $T^\sharp\mathcal{K} \subset \mathcal{K}$ for all $T \in \mathcal{T}_H$, the C^*-algebra generated by $\tau_H = \{T_\phi : \phi \in L^\infty(\mathbf{T})\}$. The operator $T^\sharp{\restriction}\mathcal{K}$ then commutes with $S^\sharp{\restriction}\mathcal{K}$; passing to $L^2(\mathbf{T})$ via the isomorphism J, we see that the operator $J^{-1}T^\sharp J$ commutes with M_z, so is of the form M_ϕ for some $\phi \in L^\infty(\mathbf{T})$. Thus, in view of the preceding paragraph, $T = T_\phi + X$, where $X^\sharp{\restriction}\mathcal{K} = 0$, i.e. $X \in \widetilde{\mathfrak{S}}_0$

[13] Here, exceptionally, dz is the arc-length measure on $\mathbf{T}$.

(Proposition 4.11). Also, $\|\phi\|_\infty = \|M_\phi\| = \|T_\phi^\sharp{\restriction}\mathcal{K}\| = \|T^\sharp{\restriction}\mathcal{K}\| \le \|T^\sharp\| = \|T\|_e \le \|T\|$, which shows that the linear map $\xi : \mathcal{T}_H \to L^\infty(\mathbf{T}) : T \mapsto \phi$ is multiplicative, because, whenever $T_1, T_2 \in \mathcal{T}_H$,

$$(T_1T_2)^\sharp{\restriction}\mathcal{K} = (T_1^\sharp{\restriction}\mathcal{K})(T_2^\sharp{\restriction}\mathcal{K}) = (JM_{\phi_1}J^{-1})(JM_{\phi_2}J^{-1}) = JM_{\phi_1\phi_2}J^{-1},$$

and so $\xi(T_1T_2) = \phi_1\phi_2 = \xi(T_1)\xi(T_2)$. Consequently, $\ker\xi$ must contain the commutator ideal $\mathrm{Com}\,\mathcal{T}_H$ of the C^*-algebra $\mathcal{T}_H$. In fact, a stronger (at first sight) condition holds: $\ker\xi$ contains not only the commutator $[T_\phi, T_\psi]$, but also the semi-commutator $[T_\phi, T_\psi) = T_{\phi\psi} - T_\phi T_\psi$, for arbitrary $\phi, \psi \in L^\infty(\mathbf{T})$:

$$(T_\phi T_\psi)^\sharp{\restriction}\mathcal{K} = (T_\phi^\sharp{\restriction}\mathcal{K})(T_\psi^\sharp{\restriction}\mathcal{K}) = (JM_\phi J^{-1})(JM_\psi J^{-1}) = JM_{\phi\psi}J^{-1} = T_{\phi\psi}^\sharp{\restriction}\mathcal{K},$$

and so

$$\xi(T_{\phi\psi} - T_\phi T_\psi) = 0. \tag{33}$$

Remark 4.12. Before going on, let us briefly discuss the relationship between various spaces of operators which have appeared above:

$$\mathfrak{S}_0 = \{T \in \mathcal{B}(H^2) : \lim_{n\to\infty} \|TS^n e_0\| = 0\} = \{T \in \mathcal{B}(H^2) : \lim_{n\to\infty} \|TS^n f\| = 0 \ \forall f \in H^2\}$$

$$\begin{aligned}\widetilde{\mathfrak{S}}_0 = \{T \in \mathcal{B}(H^2) : \mathrm{Lim}\,\|TS^n e_0\|^2 = 0\} &= \text{(sometimes)}\\ &= \{T \in \mathcal{B}(H^2) : \mathrm{Lim}\,\|TS^n f\| = 0 \ \forall f \in H^2\},\end{aligned}$$

$$\mathcal{A} = \{T \in \mathcal{B}(H^2) : T - S^*TS \in \mathtt{Comp}\},$$

$$\tau_H = \{T_\phi : \phi \in L^\infty(\mathbf{T})\}, \qquad \mathcal{T}_H = C^*\text{-alg}(\tau_H),$$

$$\mathrm{Com}\,\mathcal{T}_H = \text{the commutator ideal of } \mathcal{T}_H,$$

$$\mathtt{Comp} = \text{ the compact operators in } \mathcal{B}(H^2).$$

Consider also

$$\mathcal{E} := \{T \in \mathcal{B}(H^2) : \exists \lim_{n\to\infty} U^{*n}TS^n f, \ \forall f \in H^2\} = \{T \in \mathcal{B}(H^2) : \exists \lim_{n\to\infty} U^{*n}TS^n e_0\},$$

$$\widetilde{\mathcal{E}} := \tau_H + \widetilde{\mathfrak{S}}_0.$$

It is well-known that $\mathfrak{S}_0 \supset \mathrm{Com}\,\mathcal{T}_H = \mathfrak{S}_0 \cap \mathcal{T}_H \supset \mathtt{Comp}$ and $\mathcal{T}_H = \tau_H + \mathrm{Com}\,\mathcal{T}_H$; both inclusions are proper. Consequently,

$$\tau_H + \mathtt{Comp} \subset \tau_H + \mathrm{Com}\,\mathcal{T}_H = \mathcal{T}_H \subset \tau_H + \mathfrak{S}_0 = \mathcal{E},$$

both inclusions being proper (the last equality is easily verified). Next, $\mathfrak{S}_0$ is a proper[14] subset of $\widetilde{\mathfrak{S}}_0$, and so $\mathcal{E}$ is a proper subset of $\widetilde{\mathcal{E}}$. As for $\mathcal{A}$, we have, of

[14] To see that the inclusion is proper, take a diagonal operator with weights c_k such that $\mathrm{Lim}\,c_k = 0$ and $\lim c_k$ does not exist.

course, $\mathcal{T}_H \subset \mathcal{A}$; further, $\mathcal{A} \setminus \mathcal{E} \neq \emptyset$ (take the diagonal operator with weights $c_k = \exp(i \ln(k+1)))$ and $\widetilde{\mathfrak{S}}_0 \setminus \mathcal{A} \neq \emptyset$ (find a sequence $\{c_k\}_{k\in\mathbf{N}}$, consisting only of infinitely many zeroes and infinitely many 1's, and such that $\operatorname{Lim} c_k = 0$; then take $\operatorname{diag}(c_n)$). The author does not know if the sets $\mathcal{E} \setminus \mathcal{A}$, $\widetilde{\mathcal{E}} \setminus \mathcal{A}$, $\mathcal{A} \setminus \widetilde{\mathcal{E}}$ and $\mathfrak{S}_0 \setminus \mathcal{A}$ are empty or not. As for the Toeplitz operators on the Bergman space $A^2(\mathbf{D})$ (remember we have agreed to denote them B_ϕ throughout this chapter), the spaces

$$\tau_B = \{B_\phi : \phi \in L^\infty(\mathbf{D})\}, \qquad \mathcal{T}_B = \ C^*\text{-alg}\ (\tau_B),$$

satisfy $\texttt{Comp} \subset \mathcal{T}_B \subset \mathcal{A}$ (the first inclusion is proper, and so is probably the second) and $\tau_B \setminus \mathcal{E} \neq \emptyset$ (the operator which has served as a counterexample to $\mathcal{A} \subset \mathcal{E}$ equals in fact B_ϕ for a suitable $\phi \in L^\infty(\mathbf{D})$, cf. Example 4.2). These assertions remain in force for the Fock space $A^2(\mathbf{C})$ in place of $A^2(\mathbf{D})$ (cf. also Example 4.2). $\square$

In the last-but-one paragraph, we have almost recovered the decomposition of $\mathcal{T}_H$ and the symbol map which were mentioned at the beginning of this chapter. Let us try to carry out the same procedure for the algebra $\mathcal{A}$.

Theorem 4.13. *Let $T \in \mathcal{A}$ and suppose $T^\sharp \mathcal{K} \subset \mathcal{K}$. Then T admits a unique decomposition of the form $T = T_\phi + R$, where $\phi \in L^\infty(\mathbf{T})$, $\|\phi\|_\infty \leq \|T\|_e \leq \|T\|$, and $R \in \widetilde{\mathfrak{S}}_0$. Besides, the map $T \mapsto \phi$ is multiplicative, i.e. if $V \in \mathcal{A}$, $V^\sharp \mathcal{K} \subset \mathcal{K}$, $V = T_\psi + Q$, then $TV \mapsto \phi\psi$. Consequently, $TV - VT \in \widetilde{\mathfrak{S}}_0$.*

Proof. Repeat verbatim what was said for the case of $\mathcal{T}_H$: $T^\sharp \mathcal{K} \subset \mathcal{K} \implies J^{-1} T^\sharp J : L^2(\mathbf{T}) \to L^2(\mathbf{T})$, $T \in \mathcal{A} \implies J^{-1} T^\sharp J$ commutes with M_z, so is of the form M_ϕ, $\phi \in L^\infty(\mathbf{T})$. The operator $R := T - T_\phi$ then satisfies $R^\sharp \restriction \mathcal{K} = 0$, i.e. $R \in \widetilde{\mathfrak{S}}_0$. $\|\phi\|_\infty = \|M_\phi\| = \|T^\sharp \restriction \mathcal{K}\| \leq \|T^\sharp\| = \|T\|_e \leq \|T\|$, whence also the uniqueness assertion follows. Finally, $(TV)^\sharp \restriction \mathcal{K} = (T^\sharp \restriction \mathcal{K})(V^\sharp \restriction \mathcal{K}) = (J M_\phi J^{-1})(J M_\psi J^{-1}) = J M_{\phi\psi} J^{-1} = (VT)^\sharp \restriction \mathcal{K}$. $\square$

Thus, things go well when $T^\sharp \mathcal{K} \subset \mathcal{K}$, and this happens iff $T \in \tau_H + \widetilde{\mathfrak{S}}_0 = \widetilde{\mathcal{E}}$ (and $T \in \mathcal{A}$). In general, we have only the following weaker result.

Proposition 4.14. *Let $T \in \mathcal{A}$. Then $T = T_\phi + R$, where $\|\phi\|_\infty \leq \|T\|_e \leq \|T\|$ and $P(\mathcal{K}) R^\sharp \restriction \mathcal{K} = 0$. The decomposition is unique.*

Proof. $T \in \mathcal{A} \implies T^\sharp S^\sharp = S^\sharp T^\sharp$; denote $W = P(\mathcal{K}) T \restriction \mathcal{K}$. Then $W\mathcal{K} \subset \mathcal{K}$ and $WS^\sharp \restriction \mathcal{K} = P(\mathcal{K}) T S^\sharp \restriction \mathcal{K} = S^\sharp W$, because $S^\sharp P(\mathcal{K}) = P(\mathcal{K}) S^\sharp$ (since $\mathcal{K}$ reduces $S^\sharp$). This implies, once again, $J^{-1} W J = M_\phi$, $\phi \in L^\infty(\mathbf{T})$, and the operator $R := T - T_\phi$ must satisfy $P(\mathcal{K}) R^\sharp \restriction \mathcal{K} = 0$. The norm estimate is proved in the same way as before. $\square$

Note that, however, this map $T \mapsto \phi$ is no longer multiplicative. The operators R such that $P(\mathcal{K}) R^\sharp \restriction \mathcal{K} = 0$ admit an easy characterization: their matrices have entries which "tend to zero" along every diagonal parallel to the main diagonal.

Proposition 4.15. *Assume that $R \in \mathcal{A}$ or that* Lim *satisfies (C4). Then $P(\mathcal{K}) R^\sharp \restriction \mathcal{K} = 0$ if and only if*

$$\operatorname*{Lim}_k \langle Re_k, e_{N+k} \rangle = 0 \qquad \forall N \in \mathbf{Z}. \tag{34}$$

Proof. $P(\mathcal{K})R^\sharp\restriction\mathcal{K} = 0$ iff $\langle R^\sharp E_N, E_M\rangle = 0$ $\forall M, N \in \mathbf{Z}$. If $R \in \mathcal{A}$, this is equivalent to $\langle R^\sharp E_0, E_M\rangle = 0$ $\forall M \in \mathbf{Z}$, which is exactly (34). In the other case,

$$\langle R^\sharp E_N, E_M\rangle = \operatorname*{Lim}_k \langle Re_{N+k}, e_{M+k}\rangle = \operatorname*{Lim}_k \langle Re_k, e_{M-N+k}\rangle,$$

the last equality being a consequence of (C4). □

We close this chapter with two observations. The first concerns the Allan-Douglas localization principle [12]. Because $T^\sharp S^\sharp = S^\sharp T^\sharp$ if $T \in \mathcal{A}$, the space $\ker(S^\sharp - \mu) = \ker(S^{\sharp *} - \overline{\mu})$ is invariant under $T^\sharp$ when $T \in \mathcal{A}$, for every $\mu \in \mathbf{T}$. Denote $\mathcal{I}_\mu$ the (closed, two-sided, *-) ideal in $\mathcal{A}$ generated by $\texttt{Comp}$ and $S - \mu$. Let $\mathcal{A}_\mu$ be the C^*-algebra $\mathcal{A}/\mathcal{I}_\mu$. Similar construction may be applied to the algebras $\mathcal{T}_B$ and $\mathcal{T}_F$.

Theorem 4.16. *The mapping $W \mapsto (W + \mathcal{I}_\mu)_{\mu\in\mathbf{T}}$ from $\mathcal{A}/\texttt{Comp}$ into the C^*-direct sum $\bigoplus_{\mu\in\mathbf{T}} \mathcal{A}_\mu$ is an (isometrical) C^*-isomorphism of $\mathcal{A}/\texttt{Comp}$ onto a C^*-subalgebra of $\bigoplus_{\mu\in\mathbf{T}} \mathcal{A}_\mu$.*

The same assertion is valid for $\mathcal{T}_B$ and $\mathcal{T}_F$ in place of $\mathcal{A}$.

Proof. Follows from the general Allan-Douglas theory; for instance, apply [7, proof of Theorem 1.34–d.] with $A = \mathcal{A}/\texttt{Comp}$, $B = \{f(s) : f \in C(\mathbf{T})\}$, where $s = \pi(S)$ is the image of S in $\texttt{Calk}(H^2)$, $N = \{f(s) : f \in C(\mathbf{T}),\ f(\mu) = 0\}$, $J_N = \mathcal{I}_\mu/\texttt{Comp}$, $A_N = \mathcal{A}_\mu/\texttt{Comp}$. □

Since, as was pointed out at the beginning of this chapter, $\texttt{Comp} \subset \operatorname{Com}\mathcal{A}$, multiplicative linear functionals on $\mathcal{A}$ correspond bijectively to those on $\mathcal{A}/\texttt{Comp}$. If ϕ is one of the latter, then $\phi(s) \in \sigma(s) = \sigma_e(S) = \mathbf{T}$, and so $\phi(S-\mu) = 0$ for some $\mu \in \mathbf{T}$, i.e. $\ker\phi \supset \mathcal{I}_\mu/\texttt{Comp}$, so ϕ induces a multiplicative linear functional on $\mathcal{A}_\mu/\texttt{Comp}$ and hence on $\mathcal{A}_\mu$. On the other hand, every multiplicative linear functional on $\mathcal{A}_\mu$ yields a multiplicative linear functional on $\mathcal{A}$. Thus, the problem of existence of multiplicative linear functionals on $\mathcal{A}$ can be reduced to the same problem for $\mathcal{A}_\mu$. Unfortunately, the latter seems to be equally hopeless to solve. We conclude this small digression with a proposition which shows that it suffices to deal with $\mathcal{A}_1$.

Lemma 4.17. *For $\epsilon \in \mathbf{T}$ and f a function on $\mathbf{D}$ or $\mathbf{C}$, let $(R_\epsilon f)(z) := f(\epsilon z)$. Then $R_\epsilon^{-1} = R_{\overline{\epsilon}}$, R_ϵ is a unitary (i.e. isometric and onto) operator on $A^2(\mathbf{D}), L^\infty(\mathbf{D})$, $A^2(\mathbf{C})$ and $L^\infty(\mathbf{C})$, and the transformation $T \mapsto R_\epsilon^* T R_\epsilon$ is an isometry on both $\mathcal{B}(A^2(\mathbf{D}))$ and $\mathcal{B}(A^2(\mathbf{C}))$ which maps $\texttt{Comp}$ into $\texttt{Comp}$, $\mathcal{A}$ onto $\mathcal{A}$, and B_ϕ into $B_{R_{\overline{\epsilon}}\phi}$, resp. F_ϕ into $F_{R_{\overline{\epsilon}}\phi}$.*

Proof. Straightforward. □

Proposition 4.18. *For any μ, $\nu \in \mathbf{T}$, $\mathcal{A}_\mu \simeq \mathcal{A}_\nu$ (isometrical C^*-isomorphism).*

Proof. The transformation $T \mapsto R_\epsilon^* T R_\epsilon$ maps $\mathcal{A}$ onto $\mathcal{A}$, $\texttt{Comp}$ onto $\texttt{Comp}$, and $S - \mu$ into $\overline{\epsilon}S - \mu$; consequently, it must map $\mathcal{I}_\mu$ onto $\mathcal{I}_{\epsilon\mu}$ and $\mathcal{A}_\mu$ onto $\mathcal{A}_{\epsilon\mu}$. The map is, moreover, clearly an isometrical C^*-isomorphism. □

The second observation applies only to the one-dimensional Fock space $A^2(\mathbf{C})$, so imagine H^2 to be identified with $A^2(\mathbf{C})$ by means of the unitary operator W_F, as was remarked at the beginning of this chapter. Let $\widetilde{T}$, once again, denote the Berezin transform of the operator $T \in \mathcal{B}(A^2(\mathbf{C}))$, i.e.

$$\widetilde{T}(\lambda) = \langle Tk_\lambda, k_\lambda \rangle, \qquad k_\lambda(z) = \exp\left(\frac{\overline{\lambda} z}{2} - \frac{|\lambda|^2}{4}\right) \in A^2(\mathbf{C}),$$

and define $\widehat{T} := F_{\widetilde{T}}$, the Toeplitz operator on $A^2(\mathbf{C})$ with symbol $\widetilde{T}$. Owing to the fact that, unlike $\mathbf{D}$, the group of isometries of the Euclidean metric on $\mathbf{C}$ admits a commutative transitive subgroup, Berger and Coburn [6] have shown that

$$\widehat{T} = \int_{\mathbf{C}} W_\alpha^* T W_\alpha \, d\mu(\alpha),$$

where the integral is the weak (Gelfand-Pettis) integral, and W_α are unitary operators on $A^2(\mathbf{C})$ given by

$$(W_\alpha f)(z) = k_\alpha(z) f(z - \alpha), \qquad \alpha \in \mathbf{C}.$$

They proved the following lemma (attributed to W.Zame).

Lemma 4.19. *Let H be a separable Hilbert space, (X, ν) a finite measure space, and $F : X \to \mathcal{B}(H)$ a norm-bounded, WOT-measurable function (i.e. $x \mapsto \langle F(x)f, g\rangle$ is ν-measurable $\forall f, g \in H$), whose values are compact operators. Then the weak (Gelfand-Pettis) integral*

$$\int_X F(x) \, d\nu(x)$$

(exists and) is also a compact operator.

Proof. See [6], Lemma 12; we reproduce the proof here for convenience. First, for arbitrary $f, g \in H$, the function $\langle F(x)f, g\rangle$ is bounded and ν-measurable, so the integral $\int_X \langle F(x)f, g\rangle \, d\nu(x)$ exists and its modulus does not exceed

$$\nu(X) \, . \, \sup_X \|F(x)\| \, . \, \|f\|.\|g\|.$$

Consequently, the weak integral

$$\int_X F(x) \, d\nu(x)$$

exists and its norm is bounded by $\nu(x) \, . \, \sup_X \|F(x)\|$.

Pick a basis $\{e_j\}_{j \in \mathbf{N}}$ of H and let P_k stand for the projection onto the span of $e_0, e_1, \dots, e_k$. Denote

$$E_k = \{x \in X : \|P_k F(x) - F(x)\| < \epsilon\}.$$

Note that $\bigcup_{k \in \mathbf{N}} E_k = X$ since F takes values in `Comp`. Further, the function $x \mapsto \|P_k F(x) - F(x)\|$ is measurable, since it equals

$$\sup |\langle (P_k F(x) - F(x)) f, g\rangle|, \tag{35}$$

the supremum being taken over f, g in a countable dense subset of the unit ball of H. Consequently, the sets

$$E'_k = E_k \setminus \bigcup_{j=0}^{k-1} E_j$$

are measurable, disjoint, and their union is all of X. Choose m so large that

$$\sum_{k>m} \nu(E'_k) < \epsilon.$$

Then

$$\int_X F(x)\, d\nu(x) = \sum_{k=1}^{m} P_k \int_{E'_k} F(x)\, d\nu(x) + \\ \sum_{k=1}^{m} \int_{E'_k} [F(x) - P_k F(x)]\, d\nu(x) + \int_{\bigcup_{k>m} E'_k} F(x)\, d\nu(x).$$

Since $P_k \in \mathtt{Comp}$, the first summand belongs to $\mathtt{Comp}$, while the norms of the second and the third do not exceed $\epsilon\nu(X)$ and $\epsilon \sup_{x\in X} \|F(x)\|$, respectively. As $\epsilon > 0$ was arbitrary, the assertion of the lemma follows. $\square$

Corollary 4.20. *If $K \in \mathtt{Comp}$, then $\widehat{K} \in \mathtt{Comp}$.*

Proof. Apply the lemma to $X = \mathbf{C}$, $\nu = \mu$ (the Gaussian measure) and $F(x) = W_\alpha^* K W_\alpha$. The function $F(\alpha)$ is even WOT-continuous, since $\alpha \mapsto W_\alpha$ is SOT continuous. To prove the latter, it suffices to check that $\alpha \mapsto W_\alpha g_\lambda$ is continuous for each $\lambda \in \mathbf{C}$ (because the linear combinations of g_λ are dense in $A^2(\mathbf{C})$, cf. Lemma 2.5), which is easily verified. $\square$

Corollary 4.21. *If $T \in \mathcal{A}$, then $\widehat{T} \in \mathcal{A}$.*

Proof. We have

$$\widehat{T} - S^* \widehat{T} S = \int_{\mathbf{C}} (W_\alpha^* T W_\alpha - S^* W_\alpha^* T W_\alpha S)\, d\mu(\alpha).$$

However,

$$W_\alpha = B_{\exp(\frac{|\alpha|^2}{4} + i\mathrm{Im}(\overline{\alpha} z))}, \qquad \text{(see [6], section 3)}, \tag{36}$$

which implies $W_\alpha^* T W_\alpha \in \mathcal{A}$ whenever $T \in \mathcal{A}$. It remains to apply the lemma to $X = \mathbf{C}$, $\nu = \mu$ and $F(\alpha) = W_\alpha^* T W_\alpha - S^* W_\alpha^* T W_\alpha S$; again, F is even WOT-continuous. $\square$

The last two corollaries prompt to formulate the following

Conjecture. If $T \in \mathrm{Com}\,\mathcal{A}$, then $\widehat{T} \in \mathrm{Com}\,\mathcal{A}$.

This conjecture could be proved if we could replace $\mathcal{B}(H)$ and $\mathtt{Comp}$ in Lemma 4.19 by $\mathcal{A}$ and $\mathrm{Com}\,\mathcal{A}$, respectively. The author suspects that the lemma remains in force even if $\mathcal{B}(H)$ and `Comp` are replaced by an arbitrary C^*-algebra $\mathfrak{A}\subset \mathcal{B}(H)$ and a closed, two-sided *-ideal $\mathcal{I}$ in $\mathfrak{A}$. The above approach of W.Zame works whenever the C^*-algebra $\mathfrak{A}$ has a countable approximate identity, or if X is a separable topological space, ν a Borel measure, and the function $x \mapsto \|QF(x)\|$ is continuous for every operator $Q \in\mathfrak{A}$, $0 \leq Q \leq I$. Regarding the conjecture above, the former possibility (a countable approximate identity for $\mathcal{A}$) is highly improbable, while the latter seems quite likely indeed to take place, although I have not been able to prove it (i.e. to prove that the function

$$\alpha \mapsto \|QW_\alpha^* T W_\alpha\|, \qquad \alpha \in \mathbf{C},$$

is continuous; it is easy to see from (35) that it is lower semicontinuous, which, however, is not enough for our purposes).

If the last conjecture is true, we have a most interesting corollary:

(

Corollary 4.22.) *For all $T \in \mathcal{A}$, the operator $T - \widehat{T}$ belongs to $\mathrm{Com}\,\mathcal{A}$.*

Proof. According to the definition of $\widehat{T}$,

$$T - \widehat{T} = \int_{\mathbf{C}} W_\alpha^*[W_\alpha, T]\, d\mu(\alpha),$$

and the integrand is bounded by $2\|T\|$, WOT-continuous, and, in view of (36), belongs to $\mathrm{Com}\,\mathcal{A}$ for every $\alpha \in \mathbf{C}$. $\square$

Thus, in the abelianized algebra $\mathcal{A}_{ab} := \mathcal{A}/\mathrm{Com}\,\mathcal{A}$, the transformation $T \mapsto \widehat{T}$ would act as the identity; consequently, some insight into the structure of that algebra (multiplicative linear functionals etc.) could be gained by studying the iterated transforms $T \mapsto \widehat{T} \mapsto \widehat{\widehat{T}} \mapsto \widehat{\widehat{\widehat{T}}} \mapsto \dots$ or, which amounts to the same, of the iterated Berezin transforms $f \mapsto \widetilde{f} \mapsto \widetilde{f}^{(2)} \mapsto \widetilde{f}^{(3)} \mapsto \dots$ on $L^\infty(\mathbf{C})$. The latter is equivalent to the study of the asymptotics of the heat equation, since

$$\widetilde{f}^{(k)}(z) = u(k/2, z),$$

where $u(t,z)$ is the solution on $(0, +\infty) \times \mathbf{C}$ of the heat equation $\dfrac{\partial u}{\partial t} = 4\Delta\, u(t,z)$ with initial condition $u(0,z) = f(z)$. (See [6] again, or compute directly.) This is one of motivations for undertaking a further study of the Berezin transform, which is the objective of the subsequent chapter.

Chapter 5. THE BEREZIN TRANSFORM

Since the Berezin transform seemingly plays an important role in the theory of Toeplitz operators on the Bergman space and the Fock space, it will be studied in a greater detail in this chapter. Three main topics will be discussed. First, we investigate the boundedness of the Berezin transform on various L^p spaces. The connection of the transform with the Laplace and the Laplace-Beltrami operator, respectively, is established. Second, we prove that the fixed points of the Berezin transform are exactly the harmonic functions. Third, some ergodicity properties are discussed; the motivation for this comes from the considerations at the end of Chapter 4. The results are stated for $A^2(\mathbf{D})$ and $A^2(\mathbf{C})$ only; generalizations to other domains are likely to be possible, but seem to be closer in spirit to Riemannian geometry rather than functional analysis. We begin by recalling the definitions and basic properties.

If $f \in L^1(\mathbf{D}, d\nu)$, the *Berezin transform* of f is, by definition,

$$\widetilde{f}(w) = \langle f k_w, k_w \rangle = \int_{\mathbf{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} f(z)\, d\nu(z), \qquad w \in \mathbf{D},$$

where k_w is the normalized reproducing kernel at $w \in \mathbf{D}$:

$$k_w(z) = \frac{g_w(z)}{\|g_w\|} = \frac{1-|w|^2}{(1-\overline{w}z)^2}.$$

Note that $k_w \in L^\infty(\mathbf{D})\ \forall w \in \mathbf{D}$, so the definition makes sense.

Similarly, if $f \in L^1(\mathbf{C}^N, d\mu)$, define $\widetilde{f}$ to be

$$\widetilde{f}(w) = \langle f k_w, k_w \rangle = \int_{\mathbf{C}^N} e^{-|w-z|^2/2} f(z)\, \frac{dz}{(2\pi)^N}, \qquad w \in \mathbf{C}^N,$$

where[15]

$$k_w = \frac{g_w(z)}{\|g_w\|} = e^{\frac{\overline{w}z}{2} - \frac{|w|^2}{4}}.$$

(Since it will always be clear whether we are discussing $A^2(\mathbf{C}^N)$ or $A^2(\mathbf{D})$, no ambigiuty concerning g_w and k_w should arise.)

For typographical reasons, the Berezin transform $\widetilde{f}$ is sometimes also denoted Bf.

Proposition 5.1. *$\widetilde{f}$ is an infinitely differentiable function on $\mathbf{D}$, resp. $\mathbf{C}^N$.*

Proof. Differentiate under the integral sign. (See the proof of Proposition 5.20 for details.) $\square$

Proposition 5.2. *If f is bounded, then so is $Bf \equiv \widetilde{f}$ and $\|\widetilde{f}\|_\infty \le \|f\|_\infty$. In other words, B is a contraction in $L^\infty(\mathbf{D})$, resp. $L^\infty(\mathbf{C}^N)$.*

Proof. $|\widetilde{f}(w)| \le \|fk_w\|_2 \|k_w\|_2 \le \|f\|_\infty . \|k_w\|_2^2 = \|f\|_\infty$. $\square$

[15]Here, as usual, $\overline{w}z = \sum_{j=1}^N \overline{w}_i z_i$, $|w|^2 = \overline{w}w$, etc.

Remark 5.3. Since $f = \widetilde{f}$ when f is a constant function, the norm of B on $L^\infty(\mathbf{C}^N)$ or $L^\infty(\mathbf{D})$ is, in fact, equal to one. □

Proposition 5.4. *If $f \geq 0$, then $\widetilde{f} \geq 0$; if $f \geq g$, then $\widetilde{f} \geq \widetilde{g}$.*

Proof. B is an integral operator with positive kernel. □

We are going to show that the Berezin transform is also a contractive linear operator on other L^p spaces, provided they are taken with respect to an appropriate measure — namely, the measure which is intrinsic for the Riemannian geometry of the domain. Recall that the Lebesgue measure dz on $\mathbf{C}^N$ is (up to multiplication by a constant factor) the only measure invariant with respect to the group of the rigid motions of $\mathbf{C}^N$. Similarly, on $\mathbf{D}$, the only measure left invariant by all Möbius transformations

$$z \mapsto \epsilon \frac{z-w}{1-\overline{w}z} := \epsilon\omega_w(z) \qquad (w \in \mathbf{D}, \epsilon \in \mathbf{T}) \tag{37}$$

is the pseudo-hyperbolic measure

$$d\eta(z) := \frac{d\nu(z)}{(1-|z|^2)^2}.$$

(The invariance may be verified by direct computation.) It turns out that the Berezin transform behaves well with respect to the invariant measures.

Proposition 5.5. *(a) The mapping $B : f \mapsto \widetilde{f}$ is a contractive linear operator on each of the spaces*

$$L^p(\mathbf{D}, d\eta(z)), \qquad 1 \leq p \leq \infty.$$

(b) Similar assertion holds for $L^p(\mathbf{C}^N, dz)$, $1 \leq p \leq \infty$.

Proof. (a) Since $L^1(\mathbf{D}, d\eta) \subset L^1(\mathbf{D}, d\nu)$, the Berezin transform is defined on the former space, and

$$|\widetilde{f}(w)| = \left| \int_{\mathbf{D}} f(z) \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} \, d\nu(z) \right| \leq B(|f|)(w).$$

Thus

$$\begin{aligned} \int_{\mathbf{D}} |\widetilde{f}(w)| \, \frac{d\nu(w)}{(1-|w|^2)^2} &\leq \int_{\mathbf{D}} \left(\int_{\mathbf{D}} |f(z)| \, \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} \, d\nu(z) \right) \frac{d\nu(w)}{(1-|w|^2)^2} = \\ &= \int_{\mathbf{D}} |f(z)| \int_{\mathbf{D}} \frac{d\nu(w)}{|1-\overline{w}z|^4} \, d\nu(z) = \\ &= \int_{\mathbf{D}} |f(z)| \, . \, \langle g_w, g_w \rangle \, d\nu(z) = \int_{\mathbf{D}} |f(z)| \, \frac{d\nu(z)}{(1-|z|^2)^2}, \end{aligned}$$

the change of the order of integration being justified by the positivity of the integrand. It follows that B is a contraction on $L^1(\mathbf{D}, d\eta)$. The same is true for L^∞ (Proposition

5.2), and so the result follows from the Marcinkiewicz interpolation theorem. The proof of (b) is similar. □

The last proposition suggests that there might be some closer relationship between the Berezin transform and the Riemannian geometry on $\mathbf{D}$, resp. $\mathbf{C}^N$. This is indeed the case. Before clarifying this point, we establish an alternative formula for $\widetilde{f}$.

Proposition 5.6. *(a) In the notation of (37), for arbitrary $f \in L^1(\mathbf{D}, d\nu)$,*

$$\widetilde{f}(w) = \int_{\mathbf{D}} f(\omega_{-w}(y))\, dy.$$

(b) Similarly,

$$\widetilde{f}(w) = \int_{\mathbf{C}^N} f(y+w)\, d\mu(y)$$

for every $f \in L^1(\mathbf{C}^N, d\mu)$.

Proof. If we make an (analytic) change of coordinates

$$y = \omega_w(z), \text{ i.e. } z = \omega_{-w}(y),$$

then

$$d\nu(y) = |\omega_w'|^2 \,.\, d\nu(z) = \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\nu(z),$$

and so

$$\widetilde{f}(w) = \int_{\mathbf{D}} f(z)\, \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\nu(z) = \int_{\mathbf{D}} f(\omega_{-w}(y))\, d\nu(y)$$

as claimed. (b) is similar, only even simpler. □

Proposition 5.7. *The Berezin transform commutes with the "group of rigid motions" of $\mathbf{D}$, resp. $\mathbf{C}^N$. More precisely,*

$$B(f \circ \omega_a) = (Bf) \circ \omega_a$$

for every $f \in L^1(\mathbf{D}, d\nu)$, $a \in \mathbf{D}$, while

$$B(f \circ t_a) = (Bf) \circ t_a$$

for every $f \in L^1(\mathbf{C}^N, d\mu)$, $a \in \mathbf{C}^N$; here $t_a(z) := z - a$.

Proof. For $f \in L^1(\mathbf{D}, d\nu)$ and $a, w \in \mathbf{D}$,

$$\begin{aligned} B(f \circ \omega_a)(w) &= \int_{\mathbf{D}} f(\omega_a(\omega_{-w}(z)))\, d\nu(z) \\ &= \int_{\mathbf{D}} f(\omega_a(\omega_{-w}(\frac{1+\overline{a}w}{1+a\overline{w}} z)))\, d\nu(z) \\ &= \int_{\mathbf{D}} f(\omega_{\left\{\frac{1+a\overline{w}}{1+\overline{a}w}\omega_w(a)\right\}}(z))\, d\nu(z) \\ &= \widetilde{f}(-\frac{1+a\overline{w}}{1+\overline{a}w}\omega_w(a)) \\ &= \widetilde{f}(-\omega_{-a}(-w)) = \widetilde{f}(\omega_a(w)), \end{aligned}$$

since
$$\omega_a(\omega_{-w}(\epsilon z)) = \epsilon\frac{1+a\overline{w}}{1+\overline{a}w}\cdot\omega_{\{\overline{\epsilon}\omega_w(a)\}}(z) \qquad \text{for } \epsilon\in\mathbf{T},\ a,w\in\mathbf{D}$$
and
$$\frac{1+a\overline{w}}{1+\overline{a}w}\,\omega_{-w}(a) = \omega_{-a}(w) = -\omega_a(-w).$$
The proof for $\mathbf{C}^N$ is, once again, similar but much simpler, because the translations t_a on $\mathbf{C}^N$, unlike Möbius transformations, commute. □

The last proposition has important consequences, since operators commuting with the (Möbius or Euclidean) translations may be described explicitly. Consider the Laplace operator
$$\Delta := \prod_{j=1}^{N} 4\frac{\partial^2}{\partial z_j \partial \overline{z}_j}$$
on $\mathbf{C}^N$, and the Laplace-Beltrami operator[16]
$$\Delta_h := (1-|z|^2)^2\frac{\partial^2}{\partial z\partial\overline{z}}$$
on $\mathbf{D}$. These operators are symmetric on the subspace of $L^2(\mathbf{C}^N, dz)$ or $L^2(\mathbf{D}, d\eta)$, respectively, consisting of infinitely differentiable functions with compact support; since their coefficients are real, they can be extended to (unbounded) selfadjoint operators on the respective L^2 spaces. Besides, direct calculation reveals that they commute with the group of motions of $\mathbf{C}^N$ and $\mathbf{D}$, respectively:
$$\Delta(f\circ t_a) = (\Delta f)\circ t_a, \qquad \Delta_h(f\circ\omega_a) = (\Delta_h f)\circ\omega_a.$$
According to a fundamental work of Gelfand [17], a sort of converse also holds : every operator on $L^2(\mathbf{C}^N, dz)$ or $L^2(\mathbf{D}, d\eta)$, commuting with the group of motions, must be a function of Δ or Δ_h, respectively. In view of Proposition 5.7, this applies, in particular, to the Berezin transform — it must be a function of Δ, resp. Δ_h. This idea goes back to Berezin [5], who even exhibited an explicit formula for B (on $\mathbf{D}$) in terms of Δ_h: $\widetilde{f} = F(\Delta_h)f$ where

$$F(x) = \prod_{n=1}^{\infty}\left(1-\frac{x}{n(n+1)}\right)^{-1} = \frac{\pi x}{\sin\pi\left(\sqrt{x+\frac{1}{4}}-\frac{1}{2}\right)}. \tag{38}$$

In the case of $\mathbf{C}^N$, an explicit formula has been established by Berger and Coburn [6]:

$$\widetilde{f} = e^{\Delta/2}f, \tag{39}$$

i.e. $\widetilde{f}$ is the solution of the heat equation with the initial condition f at the time $1/2$. These formulas, although established here only for $f\in L^2(\mathbf{C}^N, dz)$ or $f\in L^2(\mathbf{D}, d\eta)$,

[16] Sometimes Δ_h is defined as four times this operator.

respectively, can be shown to (be meaningful and) hold actually for wider classes of functions — this can be done e.g. by appealing to the theory of pseudodifferential operators; we won't, however, pursue this matter further.

The spaces $L^p(\mathbf{C}^N, dz)$ and $L^p(\mathbf{D}, d\eta)$ are rather small — they don't even contain (nonzero) constant functions. A question which comes into mind is whether the Berezin transform is not actually a bounded linear operator on the spaces L^p with respect to the other natural measure — namely, $L^p(\mathbf{C}^N, d\mu)$ or $L^p(\mathbf{D}, d\nu)$, respectively. This turns out to be true whenever $p > 1$. Before presenting the proof, we are going to show how the machinery of interpolation spaces may be used to obtain a weaker result. We shall temporarily restrict our attention to $\mathbf{D}$, since most proofs work, with minor modifications, for $\mathbf{C}^N$ as well.

To prove that B is a bounded operator on $L^p(\mathbf{D}, d\nu)$, $1 < p < \infty$, it would suffice to prove this fact for $p = 1$ — since B is a contraction on $L^\infty(\mathbf{D})$, we could apply the Marcinkiewicz interpolation theorem. Unfortunately, this approach will not work.

Proposition 5.8. *B is not a bounded operator on $L^1(\mathbf{D}, d\nu)$.*

Proof. If it were, its adjoint $B^d \equiv C$,

$$(Cf)(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} f(w)\, d\nu(w), \qquad z \in \mathbf{D}, \tag{40}$$

would be a bounded operator on $L^\infty(\mathbf{D})$. It will be shown below (in course of the proof of Lemma 5.13(b)) that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-\overline{z}re^{it}|^4}\, dt = \frac{1+|z|^2r^2}{(1-|z|^2r^2)^3} = \sum_{n=0}^{\infty}(n+1)^2 r^{2n}|z|^{2n},$$

for $z \in \mathbf{D}$ and $r \in (0,1)$. Consequently,

$$\begin{aligned}
(C\mathbf{1})(z) &= \int_{\mathbf{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\nu(w) = \int_0^1 (1-r^2)^2\, \frac{1}{2\pi}\int_0^{2\pi} |1-\overline{z}re^{it}|^{-4}\, dt\, 2r\, dr \\
&= \int_0^1 (1-r^2)^2 \sum_{n=0}^{\infty}(n+1)^2 r^{2n}|z|^{2n}\, 2r\, dr \\
&= \int_0^1 \sum_{n=0}^{\infty}(n+1)^2(1-t)^2 t^n |z|^{2n}\, dt \\
&= \sum_{n=0}^{\infty} \frac{2(n+1)}{(n+2)(n+3)}|z|^{2n}.
\end{aligned}$$

As $|z| \nearrow 1$, this expression behaves (asymptotically) like $-\log(1-|z|^2)$; hence $C\mathbf{1} \notin L^\infty(\mathbf{D})$, so $C \equiv B^d$ cannot be a bounded operator on $L^\infty(\mathbf{D})$. $\square$

All the same, the above method may be exploited to prove that B is a bounded operator from $L^p(\mathbf{D}, d\nu)$ into $L^q(\mathbf{D}, d\nu)$ whenever $q < p$.

Lemma 5.9. *The integral operator C given by (40) is a contraction on $L^1(\mathbf{D}, d\nu)$ which maps $L^\infty(\mathbf{D})$ boundedly into $L^p(\mathbf{D}, d\nu)$ for every $p \in \langle 1, \infty)$. Similar assertion is valid for $\mathbf{C}^N$.*

Proof. For arbitrary $f \in L^1(\mathbf{D}, d\nu)$,

$$\begin{aligned}\int_{\mathbf{D}} |Cf(z)|\, d\nu(z) &\leq \int_{\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} |f(w)|\, d\nu(w)\, d\nu(z) \\ &= \int_{\mathbf{D}} |f(w)| \int_{\mathbf{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\nu(z)\, d\nu(w) \qquad \text{(by Fubini)} \\ &= \int_{\mathbf{D}} |f(w)| \,.\, \langle k_w, k_w \rangle \, d\nu(w) = \int_{\mathbf{D}} |f(w)|\, d\nu(w),\end{aligned}$$

so C is a contraction on $L^1(\mathbf{D}, d\nu)$. If $f \in L^\infty(\mathbf{D})$, then

$$|Cf(z)| \leq \|f\|_\infty \,.\, \int_{\mathbf{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\nu(w) = \|f\|_\infty \,.\, |C\mathbf{1}(z)|.$$

Hence, to prove the second assertion of the lemma, it suffices to check that $C\mathbf{1}$ belongs to $L^p(\mathbf{D}, d\nu)$ for each $p \in \langle 1, \infty)$. In course of the proof of the preceding proposition, we have observed that $C\mathbf{1}(z)$ behaves like $-\log(1-|z|^2)$ as $|z| \nearrow 1$, so it is enough to show that $\log(1-|z|^2) \in L^p(\mathbf{D}, d\nu)$ $\forall p \in \langle 1, \infty)$. But

$$\int_{\mathbf{D}} |\log(1-|z|^2)|^p\, d\nu(z) = \int_0^1 |\log(1-r^2)|^p\, 2r\, dr = \int_0^1 |\log(1-t)|^p\, dt = \int_0^1 |\log t|^p\, dt,$$

and, changing the variable to $y = -\log t$, this reduces to

$$\int_0^{+\infty} y^p e^{-y}\, dy = \Gamma(p+1) < +\infty.$$

The proof for $\mathbf{C}^N$ is similar. $\square$

Theorem 5.10. *(a) If $1 \leq q < p \leq \infty$, then B is a bounded operator from $L^p(\mathbf{D}, d\nu)$ into $L^q(\mathbf{D}, d\nu)$.*

(b) The same assertion holds for L^p and L^q of $(\mathbf{C}^N, d\mu)$.

Proof. We shall deal only with $\mathbf{D}$, the other case being similar. Consider the integral operator C given by (40). By the previous lemma, C is a bounded operator from $L^1(\mathbf{D}, d\nu)$ into $L^1(\mathbf{D}, d\nu)$ and from $L^\infty(\mathbf{D})$ into $L^p(\mathbf{D}, d\nu)$, $\forall p \in \langle 1, \infty)$. According to the Marcinkiewicz interpolation theorem, it must be a bounded operator from $L^q(\mathbf{D}, d\nu)$ into $L^r(\mathbf{D}, d\nu)$ $\forall r \in \langle 1, q)$ for arbitrary $q \in \langle 1, \infty \rangle$. It follows that its adjoint, which is exactly B, is a bounded map from $L^{r'}(\mathbf{D}, d\nu)$ into $L^{q'}(\mathbf{D}, d\nu)$ whenever $q' \in \langle 1, \infty \rangle$ and $r' \in (q', \infty \rangle$; as $L^{q'}(\mathbf{D}, d\nu)$ is boundedly imbedded in $L^1(\mathbf{D}, d\nu)$ for arbitrary $q' \geq 1$, we may take even $q' \in \langle 1, \infty \rangle$. Changing slightly the notation produces the assertion of the theorem. $\square$

Remark 5.11. We have proved actually a little more. Recall that the space $X_1(\mathbf{D})$ is, by definition, the class of all functions f on $\mathbf{D}$ such that $f \in L^p(\mathbf{D}, d\nu)$ for all $1 \leq p < \infty$ and

$$\|f\|_X := \sup_{1 \leq p < \infty} \frac{\|f\|_p}{p} < +\infty.$$

Equipped with the norm $\|\,.\,\|_X$, $X_1(\mathbf{D})$ becomes a Banach space (cf. [21], section 4.8). In course of the proof of Lemma 5.9, we have almost proved that C maps $L^\infty(\mathbf{D})$ into $X_1(\mathbf{D})$. Indeed, it suffices, as above, to verify that $C\mathbf{1} \in X_1(\mathbf{D})$; this is reduced to the assertion that $\log(1-|z|^2) \in X_1(\mathbf{D})$, and this in turn to $\log t \in X_1(0,1)$, i.e. to the assertion that

$$\sup_{p\geq 1} \frac{\Gamma(p+1)^{1/p}}{p} < +\infty. \tag{41}$$

But, owing to Stirling's formula,

$$\Gamma(p+1)^{1/p} \sim \frac{p}{e}\,.\,(2\pi p)^{\frac{1}{2p}},$$

so (41) is true. Thus, C is a bounded map from $L^\infty(\mathbf{D})$ into $X_1(\mathbf{D})$.

The Marcinkiewicz interpolation theorem asserts that if a linear operator T maps (boundedly) $L^\infty(\mathbf{D}) \to L^\infty(\mathbf{D})$ and $L^1(\mathbf{D}, d\nu) \to L^1(\mathbf{D}, d\nu)$, it must map $L^p(\mathbf{D}, d\nu)$ boundedly into L^p $\forall p \in (1,\infty)$. The first condition may be relaxed[17] to $L^\infty(\mathbf{D}) \to BMO(\mathbf{D})$; the space $BMO(\mathbf{D})$ is bigger than $L^\infty(\mathbf{D})$, but still lies in all $L^q(\mathbf{D}, d\nu)$, $1 \leq q < \infty$. The space X_1 is, in turn, bigger than BMO while still lying in all L^q, $1 \leq q < \infty$. If we knew that the condition $L^\infty \to BMO$ may be further relaxed to $L^\infty \to X_1$, we could conclude that C would be a bounded map from L^p into L^p $\forall p \in (1,\infty)$ — and, consequently, so would be the Berezin transform B, the adjoint of C. Whether the condition may indeed be relaxed like this seems to be an interesting unsolved problem from interpolation theory. $\square$

To prove that the Berezin transform B is actually bounded on $L^p(\mathbf{D}, d\nu)$, $1 < p < \infty$, we employ a generalization of the classical Schur test.

Proposition 5.12. *Let (X, dx) and (Y, dy) be measure spaces, $k(x,y)$ a nonnegative measurable function on $X \times Y$, $1 < p < \infty$, $q = \dfrac{p}{p-1}$. If P and Q are positive measurable functions on X and Y, respectively, and α, β positive numbers such that*

$$\int_Y k(x,y)Q(y)^{q/p}\,dy \leq \alpha P(x) \qquad dx\text{-almost everywhere on } X,$$

$$\int_X k(x,y)P(x)^{p/q}\,dx \leq \beta Q(y) \qquad dy\text{-almost everywhere on } Y,$$

then the integral operator $T : L^p(Y, dy) \to L^p(X, dx)$,

$$(Tg)(x) := \int_Y k(x,y)g(y)\,dy,$$

is bounded and $\|T\|_{p\to p} \leq \alpha^{1/q}\beta^{1/p}$.

---

[17] cf. [21], paragraph 4.10.6, and the references given therein.

Proof. Let $g \in L^p(Y, dy)$. Then

$$\int_X \left(\int_Y k(x,y)|g(y)|\, dy \right)^p dx = \int_X \left(\int_Y \left[\frac{k(x,y)}{Q(y)} |g(y)|^p \right]^{1/p} \left[k(x,y)^{1/q} Q(y)^{1/q} \right] dy \right) dx$$

$$\text{(by Hölder's inequality)} \qquad \leq \int_X \left(\int_Y \frac{k(x,y)}{Q(y)} |g(y)|^p \, dy \, . \left[\int_Y k(x,y) Q(y)^{q/p} \, dy \right]^{p/q} \right) dx$$

$$\leq \int_X \left(\int_Y \frac{k(x,y)}{Q(y)} |g(y)|^p \, dy \, . \, \alpha^{p/q} P(x)^{p/q} \right) dx$$

$$\text{(by Fubini)} \qquad = \int_Y \int_X \frac{\alpha^{p/q}}{Q(y)} |g(y)|^p \, . \, P(x)^{p/q} k(x,y) \, dx \, dy$$

$$\leq \int_Y \frac{\alpha^{p/q}}{Q(y)} |g(y)|^p \, . \, \beta Q(y) \, dy = \alpha^{p/q} \beta \, . \, \|g\|_p^p.$$

It follows that $\|Tg\|_p \leq \|T(|g|)\|_p \leq \alpha^{1/q} \beta^{1/p} \|g\|_p$, as asserted. $\square$

Lemma 5.13. *(a) If $a \in (-1, 0)$, then*

$$\int_0^1 \frac{(1-t)^a}{1-Rt} \, dt \leq \frac{1}{(-a)(a+1)} \, . \, (1-R)^a \qquad \forall R \in \langle 0, 1).$$

(b) If $a \in (-1, 2)$, then

$$\int_{\mathbf{D}} \frac{(1-|x|^2)^a}{|1-x\overline{y}|^4} \, d\nu(x) \leq C_a \, . \, (1-|y|^2)^{a-2} \qquad \forall y \in \mathbf{D},$$

where C_a is a (finite) number depending only on a.

Proof. (a) Making several changes of the variable yields

$$\begin{aligned}
\int_0^1 \frac{(1-t)^a}{1-Rt} \, dt &= \int_0^1 \frac{w^a}{(1-R)+Rw} \, dw \qquad (w = 1-t, \; dt = -dw) \\
&= -\frac{1}{a} \int_1^{+\infty} \frac{s^{1/a} \, ds}{(1-R)+Rs^{1/a}} \qquad (s = w^a, \; dw = \frac{1}{a} s^{\frac{1}{a}-1} \, ds) \\
&= c \int_1^{+\infty} \frac{ds}{R+(1-R)s^c} \qquad (u = (1-R)^{1/c} s, \; ds = (1-R)^{-1/c} \, du) \\
&= c(1-R)^{-1/c} \int_{(1-R)^{1/c}}^{+\infty} \frac{du}{R+u^c},
\end{aligned}$$

where $c = -\dfrac{1}{a} \in (1, +\infty)$; but

$$\int_1^{+\infty} \frac{du}{R+u^c} \leq \int_1^{+\infty} \frac{du}{u^c} = \frac{1}{c-1},$$

$$\int_{(1-R)^{1/c}} \frac{du}{R+u^c} \leq \int_{(1-R)^{1/c}}^{1} du \leq 1,$$

so, indeed,

$$\int_0^1 \frac{(1-t)^a}{1-Rt}\,dt \leq c(1-R)^{-1/c} \cdot \frac{c}{c-1} = \frac{1}{(-a)(a+1)}(1-R)^a.$$

(b) Let $r \in (0,1)$ and denote, for brevity, $\rho = |\overline{y}r|$ and $R = |y|^2$. By virtue of the Residue theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|1-\overline{y}re^{it}|^4}\,dt = \frac{1}{2\pi}\int_0^{2\pi} |1-\rho e^{it}|^{-4}\,dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (1-\rho e^{it})^{-2}\big(1-\frac{\rho}{e^{it}}\big)^{-2}\,dt = \frac{1}{2\pi i}\oint_{\mathbf{T}} (1-\rho z)^{-2}\big(1-\frac{\rho}{z}\big)^{-2}\,\frac{dz}{z} =$$

$$= \sum_{|z|<1} \operatorname{Res}_z \frac{z}{(1-\rho z)^2(z-\rho)^2} = \operatorname{Res}_{z=\rho} \frac{z}{(1-\rho z)^2(z-\rho)^2} =$$

$$= \left(\frac{z}{(1-\rho z)^2}\right)'\Bigg|_{z=\rho} = \frac{1+\rho z}{(1-\rho z)^3}\Bigg|_{z=\rho} = \frac{1+Rr^2}{(1-Rr^2)^3}.$$

Hence

$$\begin{aligned}
\int_{\mathbf{D}} \frac{(1-|x|^2)^a}{|1-\overline{y}x|^4}\,d\nu(x) &= \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)^a}{|1-\overline{y}re^{it}|^4}\,dt\,2r\,dr = \\
&= \int_0^1 \frac{(1-r^2)^a(1+Rr^2)}{(1-Rr^2)^3}\,2r\,dr = \\
&= \int_0^1 \frac{(1-t)^a(1+Rt)}{(1-Rt)^3}\,dt \leq \\
&\leq 2\int_0^1 \frac{(1-t)^a}{(1-Rt)^3}\,dt.
\end{aligned}$$

Find $\alpha \in (-1,0)$ and $\beta \in (0,2)$ such that $\alpha + \beta = a \in (-1,2)$. Then

$$\frac{(1-t)^a}{(1-Rt)^3} = (1-R)^{\beta-2}\left(\frac{1-R}{1-Rt}\right)^{2-\beta}\left(\frac{1-t}{1-Rt}\right)^{\beta}\frac{(1-t)^\alpha}{1-Rt} \leq (1-R)^{\beta-2}\frac{(1-t)^\alpha}{1-Rt},$$

as $\dfrac{1-t}{1-Rt}, \dfrac{1-R}{1-Rt} \in \langle 0,1\rangle$.Owing to part (a),

$$\int_0^1 \frac{(1-t)^\alpha}{1-Rt}\,dt \leq \frac{1}{(-\alpha)(\alpha+1)}(1-R)^\alpha.$$

Hence

$$2\int_0^1 \frac{(1-t)^a}{(1-Rt)^3}\,dt \leq 2\,(1-R)^{\beta-2} \cdot \frac{1}{(-\alpha)(\alpha+1)}(1-R)^\alpha = C_a \,.\, (1-R)^{a-2}$$

as asserted. $\square$

Theorem 5.14. *The Berezin transform B is a bounded operator on the spaces $L^p(\mathbf{D}, d\nu)$, $1 < p < \infty$.*

Proof. Use the Schur test (Proposition 5.12) with $P(x) = (1-|x|^2)^{-1/p}$, $Q(y) = (1-|y|^2)^{-1/q}$:

$$\int_{\mathbf{D}} \frac{(1-|x|^2)^2}{|1-\overline{x}y|^4}\,(1-|y|^2)^{-1/p}\,d\nu(y) = (1-|x|^2)^2 \int_{\mathbf{D}} \frac{(1-|y|^2)^{-1/p}}{|1-\overline{x}y|^4}\,d\nu(y) \leq$$

$$\leq (1-|x|^2)^2 \,.\, C_{-1/p}\,.\,(1-|x|^2)^{-\frac{1}{p}-2} = C_{-1/p}\,.\,P(x),$$

$$\int_{\mathbf{D}} \frac{(1-|x|^2)^2}{|1-\overline{x}y|^4}\,(1-|x|^2)^{-1/q}\,d\nu(x) \leq C_{2-\frac{1}{q}}\,.\,(1-|y|^2)^{2-\frac{1}{q}-2} = C_{2-\frac{1}{q}}\,.\,Q(y),$$

by virtue of Lemma 5.13 $\square$

Remark 5.15. The bound for the norm of B on $L^p(\mathbf{D}, d\nu)$, given by the Schur test (Proposition 5.12), is $(C_{-1/p})^{1/q}.(C_{2-\frac{1}{q}})^{1/p}$. Computing this explicitly leads to $\dfrac{2p\sqrt{p}}{\sqrt{p^2-1}}$. $\square$

Let us turn now to the second topic: determination of all functions which are invariant under the Berezin transform.

Proposition 5.16. *If a function $f \in L^1(\mathbf{D}, d\nu)$ or $L^1(\mathbf{C}^N, d\mu)$ is harmonic, then $\widetilde{f} = f$.*

Proof. If $f \in L^1(\mathbf{D}, d\nu)$ is harmonic, then so is $f \circ \omega_{-w}$; by the mean value property,

$$\widetilde{f}(w) = \int_{\mathbf{D}} f(\omega_{-w}(x))\,d\nu(x) = f(\omega_{-w}(0)) = f(w).$$

The case $f \in L^1(\mathbf{C}^N)$ is similar. $\square$

A natural question to ask is if there are other functions such that $\widetilde{f} = f$. The following two propositions suggest that the answer is probably negative.

Proposition 5.17. *(a) If $f \in L^2(\mathbf{D}, d\eta)$ and $\widetilde{f} = f$, then f is harmonic.*
(b) If $f \in L^2(\mathbf{C}^N, dz)$ and $\widetilde{f} = f$, then f is harmonic.

Proof. We shall employ the formulas (38), (39).

(b) Fix a selfadjoint Laplace operator Δ on $L^2(\mathbf{C}^N, dz)$ and let $E(\lambda)$ be its resolution of the identity[18]. Assume that $\widetilde{f} = f$; by (39), this is equivalent to $e^{\Delta/2} f = f$. Consequently

$$0 = \|(e^{\Delta/2} - I)f\|^2 = \int_{\mathbf{R}} |e^{\lambda/2} - 1|^2\,d\langle E(\lambda)f, f\rangle.$$

[18]Continuity from the right is assumed

It follows that $e^{\lambda/2} = 1 \quad d\langle E(\lambda)f, f\rangle$-almost everywhere on $\mathbf{R}$ — in other words, f belongs to the range of the projection $E(0) - E(0-)$, which is exactly $\ker \Delta$. Thus $\Delta f = 0$ and we are done.

(a) is similar, only with the function $F(\lambda)$ from (38) instead of $e^{\lambda/2}$. The formula

$$F(x)^{-1} = \frac{\sin \pi \left(\sqrt{x + \frac{1}{4}} - \frac{1}{2} \right)}{\pi x}$$

implies that, for $x \in \mathbf{R}$, $F(x) = 1$ if and only if $x = 0$. Finally, $\Delta_h f = 0$ is equivalent to $\Delta f = 0$. $\square$

Remark 5.18. The last proposition does not say too much, since the only harmonic function in $L^2(\mathbf{D}, d\eta)$ or $L^2(\mathbf{C}^N, dz)$ is constant zero. To see this e.g. for $L^2(\mathbf{D}, d\eta)$, denote

$$M(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt.$$

This is a nonnegative and nondecreasing function of r. At the same time,

$$\|f\|^2_{L^2(\mathbf{D}, d\eta)} = \int_0^1 M(r) \cdot \frac{2r}{(1-r^2)^2} \, dr < +\infty,$$

so $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$, whence $f = 0$. $\square$

Therefore it would be desirable to extend the last result to some larger space — say, at least, to $L^\infty(\mathbf{D})$ or $L^\infty(\mathbf{C}^N)$, respectively. In the Fock space setting, this can be done quite easily.

Proposition 5.19. *Assume $f \in L^\infty(\mathbf{C}^N)$, $\widetilde{f} = f$. Then f is harmonic (and, consequently, constant).*

Proof. $f = \widetilde{f}$ is locally integrable, and so determines a distribution on $\mathbf{C}^N$; since $f = \widetilde{f}$ is moreover bounded, this distribution is tempered, and we may apply the Fourier transform $\mathcal{F}$. According to the definition of $\widetilde{f}$, $\widetilde{f} = e^{-|z|^2/2} * f$ (convolution), so $\widetilde{f} = f$ implies

$$e^{-|z|^2/2}.u(z) = u(z)$$

where $u := \mathcal{F} f$. In other words,

$$\langle u, (e^{-|z|^2/2} - 1)\phi(z) \rangle = 0$$

for every $\phi \in \mathcal{S}(\mathbf{C}^N)$, the space of rapidly decreasing functions on $\mathbf{C}^N \simeq \mathbf{R}^{2N}$. Since $e^t - 1$ behaves like t when $t \to 0$, the last condition is equivalent to

$$\langle u, -|z|^2 \phi(z) \rangle = 0 \qquad \forall \phi \in \mathcal{S}(\mathbf{C}^N),$$

i.e. to $-|z|^2 u(z) = 0$. Applying $\mathcal{F}^{-1}$ gives $\Delta f = 0$ as desired. $\square$

This proof cannot be carried over verbatim to $\mathbf{D}$, since there is no analogue of the Fourier transform which would behave reasonably with respect to Δ_h. Using the method from the proof of Theorem 5.22 below, it is possible to show that $f \in L^1(\mathbf{D}, d\eta)$ is harmonic if $\widetilde{f} = f$ and Δf is Lebesgue integrable; the last condition, however, need not be a priori satisfied even for a bounded f. All that can be said is that $\Delta_h f$ must be bounded if $f = \widetilde{f} \in L^\infty(\mathbf{D})$:

Proposition 5.20. *Assume that $f \in L^\infty(\mathbf{D})$. Then $\|\Delta_h \widetilde{f}\|_\infty \leq 5\|f\|_\infty$.*

Proof. As usual, if $w = x + yi \in \mathbf{D}$, denote $\dfrac{\partial}{\partial w} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$ and $\dfrac{\partial}{\partial \overline{w}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$. A short computation yields

$$\frac{\partial}{\partial w} \frac{(1 - |w|^2)^2}{|1 - \overline{w}z|^4} = \frac{2(\overline{z} - \overline{w})(1 - \overline{w}z)(1 - |w|^2)}{|1 - \overline{w}z|^6}.$$

Because $|z - w|.|1 - \overline{w}z| = |1 - \overline{w}z|^2 . \left|\dfrac{z - w}{1 - \overline{w}z}\right| \leq |1 - \overline{w}z|^2$, it follows that

$$\left| f(z) . \frac{\partial}{\partial w} \frac{(1 - |w|^2)^2}{|1 - \overline{w}z|^4} \right| \leq \|f\|_\infty . \frac{2(1 - |w|^2)}{|1 - \overline{w}z|^4} \leq \frac{2\|f\|_\infty}{(1 - |w|)^4}.$$

This is uniformly bounded when w runs through a small vicinity of an arbitrary point $w_0 \in \mathbf{D}$. Consequently, it is legal to differentiate under the integral sign in the formula defining $\widetilde{f}$, which gives

$$\frac{\partial \widetilde{f}}{\partial w} = \int_{\mathbf{D}} f(z) . \frac{2(\overline{z} - \overline{w})(1 - \overline{w}z)(1 - |w|^2)}{|1 - \overline{w}z|^6} \, d\nu(z).$$

Going through the same procedure once again, we see that

$$\frac{\partial^2 \widetilde{f}}{\partial w \partial \overline{w}} = \int_{\mathbf{D}} K(w, z) f(z) \, d\nu(z),$$

where the kernel is

$$\begin{aligned} K(z, w) =& \frac{\partial^2}{\partial w \partial \overline{w}} \frac{(1 - |w|^2)^2}{|1 - \overline{w}z|^4} = \\ =& (4|w|^2 - 2) \frac{(1 - |w|^2)^2}{|1 - \overline{w}z|^4} + 4\frac{1 - |w|^2}{|1 - \overline{w}z|^4} - 4\frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^6}. \end{aligned}$$

Recalling the formulas

$$(1 - \overline{w}z)^{-2} = \sum_{n=0}^{\infty} (n + 1)\overline{w}^n z^n \qquad \text{and} \qquad (1 - \overline{w}z)^{-3} = \sum_{n=0}^{\infty} \frac{(n + 1)(n + 2)}{2} \overline{w}^n z^n$$

and integrating term by term[19], we get

$$\int_{\mathbf{D}} \frac{d\nu(z)}{|1 - \overline{w}z|^4} = (1 - |w|^2)^{-2}$$

[19] This is legal (since the series converge uniformly on compact subsets of $\mathbf{D}$) if we integrate over $R\mathbf{D}$, $0 < R < 1$; doing so and taking limits for $R \nearrow 1$ leads to the desired formulas.

and

$$4\int_{\mathbf{D}} \frac{1-|z|^2}{|1-\overline{w}z|^6}\, d\nu(z) = 2(1-|w|^2)^{-3},$$

respectively. Consequently,

$$\left|\frac{\partial^2 \widetilde{f}}{\partial w \partial \overline{w}}\right| \le \|f\|_\infty \cdot \left[(4|w|^2-2) + \frac{4}{1-|w|^2} + \frac{2}{(1-|w|^2)^2}\right],$$

whence

$$|\Delta_h \widetilde{f}(w)| \le \|f\|_\infty \cdot \big[(4|w|^2-2)(1-|w|^2)^2 + 4(1-|w|^2) + 2\big].$$

For $w \in \mathbf{D}$, the maximum of the bracketed term is approximately $4.439\dots < 5$; thus

$$\|\Delta_h \widetilde{f}\|_\infty \le 5\|f\|_\infty$$

as claimed. $\square$

Using this proposition, a proof may be given that f is harmonic whenever $\widetilde{f} = f \in L^\infty(\mathbf{D})$. Since a lot technicialities seems to be necessary, only its sketch will be outlined here. Denote

$$\mathcal{M} = \{f \in L^\infty(\mathbf{D}) : f = \widetilde{f}\}.$$

This is a closed subspace of $L^\infty(\mathbf{D})$. The last proposition shows that $A := \Delta_h \restriction \mathcal{M}$ is a bounded operator on $\mathcal{M}$, $\|A\| \le 5$. Since

$$G(z) := \prod_{n=1}^\infty \left(1 - \frac{z}{n(n+1)}\right) = \frac{\sin \pi \left(\sqrt{z+\frac14} - \frac12\right)}{\pi z}$$

is an entire function, it is possible to define $G(A)$ and, owing to the spectral mapping theorem,

$$\sigma(G(A)) = G(\sigma(A)).$$

Assuming[20] that the formula (38) is valid for f and thinking for some time leads to the conclusion that $G(\Delta_h)$, whenever it is defined, must be the inverse of the Berezin transform. Consequently, $G(A) = I$ and $G(\sigma(A)) = \{1\}$. Next, show that $z = 0$ is the only zero of $G(z) - 1$ in the disc $|z| \le 5$; this implies $\sigma(A) = \{0\}$. Let $H(z) := \dfrac{G(z)-1}{z}$ and $C := H(A)$. C is clearly bounded; $\sigma(C) = \{G'(0)\} = \{-1\}$, so C is invertible; finally,

$$CA = G(A) - I = 0,$$

so $A = 0$, i.e. $\mathcal{M} \subset \ker \Delta_h$. $\square$

The last topic we wanted to discuss were ergodicity properties of the Berezin transform. Once again, situation is quite transparent when we restrict our attention

[20] It seems necessary to resort to the theory of pseudodifferential operators in order to prove this.

to $L^2(\mathbf{C}^N, dz)$ and $L^2(\mathbf{D}, d\eta)$, but gets complicated if we want to deal with wider function classes.

Proposition 5.21. *As an operator on $L^2(\mathbf{C}^N, dz)$ or $L^2(\mathbf{D}, d\eta)$, $B^n \to 0$ in the strong operator topology.*

Proof. We shall deal only with $\mathbf{C}^N$, the proof for $\mathbf{D}$ being similar. Since B is a contraction of the form $B = e^{\Delta/2}$ for a selfadjoint operator Δ, its spectrum must be contained in $\langle 0, 1\rangle$. Denote, once again, $E(\lambda)$ the resolution of the identity for the (selfadjoint) operator B. Then

$$\|B^n f\|^2 = \int_{\langle 0,1\rangle} |\lambda^n|^2 \, d\langle E(\lambda) f, f\rangle.$$

According to the Lebesgue monotone convergence theorem, this tends to

$$\|I - E(1-))f\|^2 = \|P_{\ker(B-I)} f\|^2.$$

But it follows from Proposition 5.17 and Remark 5.18 that $\ker(B-I) = \{0\}$, whence $\|B^n f\| \to 0$ as claimed. $\square$

It is easy to see that this simple behaviour does not persist when we consider B on $L^\infty(\mathbf{C}^N)$ or $L^\infty(\mathbf{D})$. For example, take $f \in L^\infty(\mathbf{C})$,

$$f(z) = \begin{cases} 1 & \text{if } \operatorname{Re} z > 0, \\ -1 & \text{if } \operatorname{Re} z < 0. \end{cases}$$

A short computation shows that

$$(B^n f)(z) = \widetilde{f}(z/\sqrt{n}),$$

so $(B^n f)(z) \to \widetilde{f}(0) = 0$ as $n \to \infty$ for all $z \in \mathbf{C}$. However, the convergence cannot be uniform since, for every $n \in \mathbf{N}$,

$$\lim_{x\to+\infty} (B^n f)(x + yi) = \lim_{x\to+\infty} \widetilde{f}(x + yi) = +1.$$

All the same, the uniform convergence can be established in some particular cases. If f is harmonic, then $Bf = f$ (Proposition 5.16), so trivially $B^n f \rightrightarrows f$. What about subharmonic f?

Theorem 5.22. *Assume that $f \in L^1(\mathbf{D}, d\nu)$ is a real-valued subharmonic function on $\mathbf{D}$ which admits an integrable harmonic majorant (i.e. there exists a function $v \in L^1(\mathbf{D}, d\nu)$ harmonic on $\mathbf{D}$ and such that $v(x) \geq f(x) \quad \forall x \in \mathbf{D}$). Then $B^n f \nearrow u$, the least harmonic majorant of f.*

Proof. According to a theorem of Frostman ([19], Theorem 5.25), there exists a positive Borel measure κ on $\mathbf{D}$ such that

$$f(w) = u(w) + \frac{1}{4} \int_{\mathbf{D}} \ln |\omega_w(x)|^2 \, d\kappa(x) \qquad \forall w \in \mathbf{D}.$$

(When f is twice continuously differentiable, $d\kappa(x) = \Delta f(x)\, d\nu(x)$; $\pi\, d\kappa$ is called the *Riesz measure* of f.) Write, for brevity, $g(x) = \ln |x|^2$. Since $|\omega_w(x)| = |\omega_x(w)|$, we have

$$f(w) = u(w) + \frac{1}{4}\int_{\mathbf{D}} g \circ \omega_x(w)\, d\kappa(x).$$

Hence

$$\widetilde{f}(z) = \widetilde{u}(z) + \int_{\mathbf{D}} \frac{(1-|z|^2)^2}{|1-\overline{w}z|^4} \cdot \frac{1}{4}\int_{\mathbf{D}} g \circ \omega_x(w)\, d\kappa(x)\, d\nu(w).$$

Since $f \le u \le v$ and $f, v \in L^1(\mathbf{D}, d\nu)$, we have $u \in L^1(\mathbf{D}, d\nu)$, so $\widetilde{u} = u$ in view of Proposition 5.16. Because the integrand is nonpositive, we may interchange the order of integration, which gives

$$\widetilde{f}(z) = u(z) + \frac{1}{4}\int_{\mathbf{D}} B(g \circ \omega_x)(z)\, d\kappa(x). \tag{42}$$

Proceeding by induction, we obtain

$$(B^n f)(z) = u(z) + \frac{1}{4}\int_{\mathbf{D}} B^n(g \circ \omega_x)(z)\, d\kappa(x).$$

Assume that

$$B^n(g \circ \omega_x)(z) \nearrow 0 \qquad \text{as } n \to \infty, \text{ for all } x, z \in \mathbf{D}. \tag{43}$$

Because κ is a positive measure, we may apply the Lebesgue monotone convergence theorem to conclude that

$$B^n f \nearrow u \qquad \text{as } n \to \infty$$

and the proof of the theorem is complete.

It remains to prove (43). Since $B^n(g \circ \omega_x) = (B^n g) \circ \omega_x$ (cf. Proposition 5.7), it suffices to show that $B^n g \nearrow 0$. By definition, $g(x) = \ln |x|^2$, while direct computation reveals that

$$\widetilde{g}(x) = |x|^2 - 1.$$

It follows that $g \le \widetilde{g}$. By Proposition 5.4, this implies $B^k g \le B^{k+1} g \quad \forall k \in \mathbf{N}$, so

$$g \le Bg \le B^2 g \le B^3 g \le \cdots \le 0.$$

Hence a limit $\psi(x) := \lim\limits_{n\to\infty} (B^n g)(x)$ must exist, $\psi \le 0$, and, owing to the Lebesgue monotone convergence theorem, $\widetilde{\psi} = \psi$. We claim that $\psi \equiv 0$. Assume the contrary. Because $0 \ge \psi(x) \ge \widetilde{g}(x) = |x|^2 - 1$, we have $\lim\limits_{|x|\to 1} \psi(x) = 0$; consequently, ψ must attain its infimum at some point $y \in \mathbf{D}$ — suppose (replacing ψ by $\psi \circ \omega_y$ otherwise) that $y = 0$. Then

$$\psi(0) = \widetilde{\psi}(0) = \int_{\mathbf{D}} \psi(x)\, d\nu(x) > \inf_{\mathbf{D}} \psi(x) \,.\, \int_{\mathbf{D}} d\nu(x) = \psi(0)$$

— a contradiction. $\square$

Remark 5.23. If f is real-valued, subharmonic and $f \in L^2(\mathbf{D}, d\eta)$, we may proceed a little more quickly. The subharmonicity implies that

$$\widetilde{f}(w) = \int_{\mathbf{D}} f(\omega_{-w}(y))\, d\nu(y) \geq f(\omega_{-w}(0)) = f(w),$$

i.e. $\widetilde{f} \geq f$. Further, B commutes with Δ_h, whence $\Delta_h \widetilde{f} = B(\Delta_h f) \geq 0$ since $\Delta_h f \geq 0$; in other words, $\widetilde{f}$ is also subharmonic. Proceeding by induction, we obtain a nondecreasing sequence $\{B^k f\}_{k \in \mathbf{N}}$ of subharmonic functions. Their limit ψ is either identically $+\infty$, or is a subharmonic function satisfying $\widetilde{\psi} = \psi$. Since $\psi \in L^2(\mathbf{D}, d\eta)$, the former case cannot occur; further,

$$\psi(0) = \widetilde{\psi}(0) = \int_{\mathbf{D}} \psi(y)\, d\nu(y),$$

and so ψ is actually harmonic; hence, it is a harmonic majorant of f. If h is another harmonic majorant of f, then $f \leq h$ implies $B^n f \leq B^n h = h$, whence also $\psi \leq h$; consequently, ψ is the least harmonic majorant of f, and we are done.

Observe that, although there is no nonzero harmonic function in $L^2(\mathbf{D}, d\eta)$, there are plenty subharmonic ones. The functions $B^n g$, $g(z) = \ln |z|^2$, $n \in \mathbf{N}$, serve as an example:

$$\int_{\mathbf{D}} |g(x)|^2\, d\nu(x) = \int_0^1 \left(\frac{\ln t}{1-t} \right)^2 dt = \int_0^1 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t^{m+n} \ln^2 t\, dt =$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{2}{(m+n+1)^3} = \sum_{k=0}^{\infty} \frac{2}{(k+1)^2} = \frac{\pi^2}{3} < +\infty.$$

$\square$

In fact, another proof of Theorem 5.22 may be given on the lines of Remark 5.23 — the assumption $f \in \mathrm{L}^2(\mathbf{D}, d\nu)$ was used only to infer that Bf is subharmonic when f is. Using the formula (42), this fact may be shown to hold in general.

Theorem 5.24. *Assume $f \in L^1(\mathbf{D}, d\nu)$ is a real-valued subharmonic function on $\mathbf{D}$ which admits an integrable harmonic majorant v. Then the functions $B^n f$ are subharmonic, $\forall n \in \mathbf{N}$.*

Proof. Let $0 < R < 1$. Owing to the formula (42),

$$\frac{1}{2\pi} \int_0^{2\pi} \widetilde{f}(Re^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it})\, dt + \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{4} \int_{\mathbf{D}} B(g \circ \omega_x)(Re^{it})\, d\kappa(x)\, dt.$$

Because the second integrand is nonpositive, we may interchange the order of integration; consequently,

$$\frac{1}{2\pi} \int_0^{2\pi} \widetilde{f}(Re^{it})\, dt = u(0) + \frac{1}{4} \int_{\mathbf{D}} \left(\frac{1}{2\pi} \int_0^{2\pi} B(g \circ \omega_x)(Re^{it})\, dt \right) d\kappa(x).$$

The function $B(g \circ \omega_x)(z) = |\omega_x(z)|^2 - 1$ is a subharmonic function (of z), which implies

$$\frac{1}{2\pi}\int_0^{2\pi} B(g \circ \omega_x)(Re^{it})\, dt \geq B(g \circ \omega_x)(0).$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(Re^{it})\, dt \geq u(0) + \frac{1}{4}\int_{\mathbf{D}} B(g \circ \omega_x)(0)\, d\kappa(x) = \widetilde{f}(0)$$

for every $R \in (0,1)$. The same procedure, of course, may be carried out for the functions $f \circ \omega_a$, $a \in \mathbf{D}$. It follows that $\widetilde{f}$ satisfies the sub-mean value property, and therefore is subharmonic on $\mathbf{D}$.

Because $f \leq v$, we have also $\widetilde{f} \leq \widetilde{v} = v$ (Propositions 5.4 and 5.16), so $\widetilde{f}$ also has an integrable harmonic majorant. Consequently, we may proceed by induction, and the theorem follows. $\square$

Given a bounded real-valued subharmonic function on $\mathbf{D}$, the boundary values of its least harmonic majorant can be described explicitly. A proof of this fact is included below, since it seems to be missing in textbooks on potential theory.

Proposition 5.25. *Suppose ϕ is a bounded real-valued subharmonic function on $\mathbf{D}$. Define ϕ on $\mathbf{T}$ by*

$$\phi(\epsilon) = \limsup_{r \nearrow 1} \phi(r\epsilon), \qquad \epsilon \in \mathbf{T}, \tag{44}$$

and let ψ be the Poisson extension of $\phi\restriction\mathbf{T}$ into the interior of $\mathbf{D}$. Then ψ is the least harmonic majorant of ϕ.

Proof. Let u be the least harmonic majorant of ϕ. Except for ϵ in a set of zero (arc-length) measure, we have

$$\lim_{r \nearrow 1} u(r\epsilon) \geq \limsup_{r \nearrow 1} \phi(r\epsilon) = \phi(\epsilon) = \lim_{r \nearrow 1} \psi(r\epsilon).$$

It follows that the bounded harmonic function $u - \psi$ has nonnegative radial limits a.e. on $\mathbf{T}$; hence, $u \geq \psi$ on $\mathbf{D}$. Let us show that also $u \leq \psi$. Because (sub)harmonicity is invariant under Möbius transformations, it suffices to show that $\psi(0) \geq u(0)$. Without loss of generality, we may assume $\phi \leq 0$; hence also $u \leq 0$, $\psi \leq 0$. Applying the Fatou lemma to the functions $t \mapsto \phi(re^{it})$, we see that

$$\frac{1}{2\pi}\int_0^{2\pi} \limsup_{r \nearrow 1} \phi(re^{it})\, dt \geq \limsup_{r \nearrow 1} \frac{1}{2\pi}\int_0^{2\pi} \phi(re^{it})\, dt.$$

The left-hand side is, by definition, $\psi(0)$, whereas the lim sup on the right-hand side may be replaced either by lim or by sup and equals $u(0)$. $\square$

Remark 5.26. The last proposition may be compared with Theorem 3.11 of [19], which asserts the same, but with (44) replaced by

$$\phi(\epsilon) = \limsup_{\mathbf{D} \ni z \to \epsilon} \phi(z)$$

The proof of our version seems to be more elementary. $\square$

The proofs of Theorem 5.22 as well as of Remark 5.23 may be carried over to the Fock space setting.

Theorem 5.27. *If f is a bounded real-valued subharmonic function on $\mathbf{C}^N$, then $B^n f \nearrow u$, the least harmonic majorant of f.*

Since the only bounded harmonic functions on $\mathbf{C}^N$ are the constant ones, necessarily

$$u(z) = \sup_{x \in \mathbf{C}^N} f(x) \qquad \forall z \in \mathbf{C}^N.$$

As a matter of fact, an analogue of Proposition 5.25 remains in force: the above supremum coincides with the radial limit of f on almost all half-lines emanating from the origin[21].

Remark 5.28. For $N = 1$, the statement of the theorem is trivial, since there are no bounded subharmonic functions on $\mathbf{C}$ but constant ones. However, nontrivial bounded subharmonic functions exist when $N \geq 2$; as an example, take $\max\{-1, |z|^{2-2N}\}$. $\square$

We conclude this chapter with an application of Theorems 5.22 and 5.27 Recall that

$$V(\mathbf{D}) := \{f \in L^\infty(\mathbf{D}) : \operatorname{ess\,lim}_{|z| \nearrow 1} f(z) = 0\}$$

and similarly for $\mathbf{C}^N$.

Proposition 5.29. *If (a) $f \in V(\mathbf{D})$ or (b) $f \in V(\mathbf{C}^N)$, then $B^n f \rightrightarrows 0$.*

Proof. (a) First observe that it suffices to consider $f \leq 0$ since B is linear. Next, it suffices to consider $f \in \mathcal{D}(\mathbf{D})$, because B is a contraction and $\mathcal{D}(\mathbf{D})$ is dense in $V(\mathbf{D})$. So suppose $f \leq 0$ and $\operatorname{supp} f \subset R\mathbf{D}$, $R \in (0, 1)$. Define the function F on $\langle 0, 1\rangle$ as follows:

$$F(t) = -\|f\|_\infty \qquad \text{if } 0 \leq t \leq R,$$

$$F(1) = 0,$$

$$F(t) \text{ is linear on } \langle R, 1\rangle,$$

and set $\phi(z) = F(|z|)$, $z \in \overline{\mathbf{D}}$. The function ϕ is subharmonic, its least harmonic majorant being constant zero. By virtue of Theorem 5.22, $B^n\phi \nearrow 0$; since $\phi = 0$ on $\mathbf{T}$, Dini's theorem forces even $B^n\phi \rightrightarrows 0$. But $\phi \leq f \leq 0$, hence $B^k\phi \leq B^k f \leq 0$, and so $B^n f \rightrightarrows 0$ as well.

(b) The proof is easier this time, since an explicit formula for $B^n f$ is well-known from the theory of the heat equation:

$$(B^n f)(z) = (2\pi n)^{-N/2} \cdot \int_{\mathbf{C}^N} \exp\left(-\frac{|z-x|^2}{2n}\right) \cdot f(x)\, dx.$$

[21]Cf. [19], Theorem 3.21.

Reasoning as above, it suffices to consider $f \geq 0$ and $\operatorname{supp} f \subset R\mathbf{D}$, $R \in (0, +\infty)$. In that case,

$$|(B^n f)(z)| \leq \frac{\|f\|_\infty}{(2\pi n)^{N/2}} \cdot \int_{|x| \leq R} dx = \frac{\gamma_N R^{2N} \|f\|_\infty}{(2\pi n)^{N/2}} \qquad \forall z \in \mathbf{C}^N,$$

γ_N being the volume of the unit ball in $\mathbf{C}^N$. Letting $n \to \infty$ yields $\|B^n f\|_\infty \to 0$ as claimed.

Corollary 5.30. *Suppose $f \in C(\overline{\mathbf{D}})$. Then $B^n f \rightrightarrows h$, the harmonic function whose boundary values coincide with $f \restriction \mathbf{T}$.*

Proof. Because $f \in C(\overline{\mathbf{D}})$, $f \restriction \mathbf{T} \in C(\mathbf{T})$, hence also $h \in C(\overline{\mathbf{D}})$ and $f - h \in V(\mathbf{D})$. It follows that $B^n(f - h) \rightrightarrows 0$. But $Bh = h$, so $B^n f \rightrightarrows h$.

CONCLUSIONS FOR FURTHER DEVELOPMENT IN THE AREA

The results above lead to some challenging questions. It is still unclear what is the norm closure of the set of all Toeplitz operators on a Bergman-type space (Chapter 2); likewise, it is an open problem whether the C^*-algebra generated by them is all of $\mathcal{A}$ or not, and whether $\mathcal{A}(U)$ and $\mathcal{A}(S)$ are isomorphic. One would also like to generalize Theorem 3.10 (or 3.17) to domains of higher dimensions; perhaps $\mathcal{A}(U)$ should play the "universal" role which $\mathcal{A}(S)$ plays for $\Omega \subset \mathbf{C}$ (cf. Remark 3.26). Many interesting problems are also encountered in Chapter 4 (the question of existence of some kind of Toeplitz calculus still remains open in general) and Chapter 5 (fixed points and ergodicity properties of the Berezin transform acting on L^∞, for example; Proposition 5.20 looks promising from this point of view).

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LIST OF NOTATION

The numbers etc. refer to the place where the symbol is defined. Since there are paragraphs which are not numbered, references like "3.30 & before" are used to indicate that the symbol first occurs between the items 3.29 and 3.30.

$A^p(\Omega)$	the Bergman ($\Omega \subset \mathbf{C}^N$) or the Fock ($\Omega = \mathbf{C}^N$) space
$\mathcal{A}(M)$	the $\mathcal{A}$-algebra corresponding to an operator M (3.30 & before)
$\mathcal{A}^\sharp(M)$	the $\mathcal{A}^\sharp$-algebra corresponding to M (3.30 & before)
$\mathcal{A}$	an abbreviation for $\mathcal{A}(S)$ (3.4)
$\mathcal{A}_\mu$	the localization of $\mathcal{A}$ at $\mu \in \mathbf{T}$ (4.16)
$\mathcal{A}_{ab}$	$= \mathcal{A}/\operatorname{Com}\mathcal{A}$
$\mathfrak{B}$	the set of all accessible boundary elements (3.12)
B	the Berezin transform, $Bf \equiv \widetilde{f}$
B_ϕ	$= W_B^* T_\phi W_B$ (the very end of Chapter 3)
$BCESV$	a function space, see 3.25 & before
$\mathcal{B}(H)$	the space of all bounded linear operators on a Hilbert space H
BMO	the space of all functions of bounded mean oscillation (sometimes also called the John-Nirenberg space, and denoted JN)
$\mathbf{C}$	the complex plane
$c_n(f)$	see 1.3, 1.7
$\mathtt{Comp}$, $\mathtt{Comp}(H)$	the compact operators (on a Hilbert space H)
$\operatorname{clos} X$	closure of a set $X \subset \mathcal{B}(H)$ in the operator norm topology
$C(\overline{\Omega})$	the space of functions continuous on the closure $\overline{\Omega}$ of a domain $\Omega \subset \mathbf{C}^N$
$\mathtt{Calk}$, $\mathtt{Calk}(H)$	the Calkin algebra (on a Hilbert space H)
$\operatorname{Com} A$	the commutator ideal of an algebra A
$C^*\text{-alg}\, X$	the smallest C^*-algebra containing a set $X \subset \mathcal{B}(H)$
C	the adjoint of B (5.8)
C_a	a constant (5.13)
^{d}	X^d, the dual of a Banach space X; T^d, the dual of an operator T
$\mathbf{D}$	the unit disc
dz	the Lebesgue measure on $\mathbf{C}^N$, $N \geq 1$
$d\nu(z)$	$= \dfrac{1}{\pi}\, dz$, a measure on $\mathbf{D}$
$d\mu(z)$	the Gaussian measure on $\mathbf{C}^N$
$d\kappa(z)$	the Riesz measure of a subharmonic function (5.21)
$\operatorname{diag}(c_n)$	the diagonal operator with weight sequence $\{c_n\}$
dist	the (euclidean) distance
$d(x, y)$	the pseudohyperbolic distance of $x, y \in \mathbf{D}$
$D_h(\lambda, R)$	the pseudohyperbolic disc with center λ and radius R
$\mathcal{D}(\Omega)$	the space of all infinitely differentiable functions the supports of which are compact subsets of $\Omega \subseteq \mathbf{C}^N$
ess lim	the essential limit; similarly ess sup
ESV	a function space, see 3.25 & before

$\{E_N\}$	an orthonormal basis in $\mathcal{H}$ and $\mathcal{K}$ (4.8)
$\mathcal{E}, \widetilde{\mathcal{E}}$	spaces of operators, see 4.12
$\{e_j\}$	orthonormal bases in various Hilbert spaces
$E(\lambda)$	resolution of the identity of a selfadjoint operator (continuity from the right is assumed)
F_ϕ	$= W_F^* T_\phi W_F$ (the very end of Chapter 3)
$\mathcal{F}$	the Fourier transform
g_λ	the reproducing kernel (or evaluation functional) at λ, for various spaces ($A^2(\Omega)$, $A^2(\mathbf{C}^N)$, $H^2(\rho)$)
$g_{(m,a)}$	the functional evaluating the m-th derivative at λ, on $A^2(\mathbf{D})$ (before Lemma 2.6)
$\mathbf{G}$	the Gauss sphere, $\mathbf{C} \cup \{\infty\}$
$g(x)$	$= \ln \lvert x\rvert^2$ at the end of Chapter 5
$H^\infty(\Omega)$	the space of bounded analytic functions on Ω
H_ϕ	the Hankel operator with symbol ϕ
H^2	the Hardy space (on the unit circle)
H^2_-	$= L^2(\mathbf{T}) \ominus H^2$
$H^2(\rho)$	Hilbert spaces, see before 3.17
$\mathcal{H}$	a subspace of $\mathcal{L}$, see 4.8
iff	if and only if
ind	the Fredholm index
$\mathcal{I}_\mu$	the local ideal, see 4.16
J	1) the diagonal operator $J = \operatorname{diag}(-1)^n$ 2) an isometry of H^2 onto $\mathcal{H}$, see 4.8
k_λ	normalized evaluation functionals, $k_\lambda = \dfrac{g_\lambda}{\lVert g_\lambda\rVert}$
$\mathcal{K}$	a subspace of $\mathcal{L}$, see 4.8 ff.
$L^p, L^p(\Omega), L^p(\Omega, d\rho)$	Lebesgue spaces ($1 \le p \le \infty$); if $d\rho$ is omitted, the Lebesgue measure is understood
$L^2(\rho), L^\infty(\rho)$	function spaces, see before 3.20
$l^2 = l^2(\mathbf{N})$	the space of all square-summable sequences of complex numbers, endowed with the usual Hilbert space structure
$l^\infty = l^\infty(\mathbf{N})$	the space of all bounded sequences of complex numbers, with supremum norm
$\mathcal{L}_\infty$	a subset of $\mathtt{Calk}(L^2(\mathbf{T}))$, see 3.32
$\mathcal{L}', \mathcal{L}'', \mathcal{L}$	spaces used to construct the Calkin representation (see before 4.3) (exceptionally, the primes do *not* denote commutants)
Lim	a Banach limit, see Chapter 4
M_ϕ	the operator of multiplication by a $\phi \in L^\infty$
$\mathbf{N}$	$= \{0, 1, 2, \dots\}$
$\widehat{n}$	the functional on l^∞: $\{x_k\}_{k\in\mathbf{N}} \mapsto x_n$
P_+	the orthogonal projection of L^2 onto H^2 or A^2, in various settings
QC	the quasicontinuous functions on $\mathbf{T}$
$\operatorname{Ran} T$	the range of a mapping T
$\mathbf{R}$	the set of all real numbers

R_ϵ	a composition operator, see 4.17
supp	the support (of a function or a measure)
S	the unilateral (forward) shift on H^2
$\mathcal{S}$	the unit sphere in $\mathbf{C}^N$
$\mathcal{S}(\mathbf{C}^N)$	the Schwarz space of rapidly decreasing functions
SOT	the strong operator topology
$\mathfrak{S}_0, \widetilde{\mathfrak{S}}_0$	spaces of operators, see page 51 and before 4.11, respectively
$\mathbf{T}$	the unit circle in $\mathbf{C}$
T_ϕ	the Toeplitz operator with symbol ϕ
$\mathcal{T}$	$= \{T_\phi : \phi \in L^\infty(\mathbf{D})\}$, a subset of $\mathcal{B}(A^2(\mathbf{D}))$
$\mathcal{T}_1$	$= \{T_\phi : \phi \in \mathcal{D}(\mathbf{D})\}$, a subset of $\mathcal{B}(A^2(\mathbf{D}))$
$\mathcal{T}_2$	$= \{T_\phi : \phi \in C(\overline{\mathbf{D}})\}$, a subset of $\mathcal{B}(A^2(\mathbf{D}))$
$\texttt{Trace}$	the ideal of trace-class operators
$\mathrm{Tr}(T)$	the trace of an operator T
$T_{(m,n,a)}$	operators on $A^2(\mathbf{D})$, see before 2.6
$\mathcal{T}_B, \mathcal{T}_F, \mathcal{T}_H$	the C^*-algebras generated by all Toeplitz operators with bounded symbols on $A^2(\mathbf{D})$, $A^2(\mathbf{C}^N)$, H^2
t_a	the translation operator, $t_a(z) = z - a, \quad a, z \in \mathbf{C}^N$
U	the bilateral (forward) shift on $L^2(\mathbf{T})$
$V(\mathbf{D})$	$= \{\phi \in L^\infty(\mathbf{D}) : \text{ess} \lim_{\lvert z\rvert \nearrow 1} \phi(z) = 0\}$
$V(\mathbf{C}^N)$	$= \{\phi \in L^\infty(\mathbf{C}^N) : \text{ess} \lim_{\lvert z\rvert \nearrow +\infty} \phi(z) = 0\}$
WOT	the weak operator topology
W_B, W_F	the unitary operators of H^2 onto $A^2(\mathbf{D})$, $A^2(\mathbf{C}^N)$ (see the very end of Chapter 3)
$X_1(\mathbf{D}), X_1(0,1)$	Banach function spaces, see 5.11
$\mathbf{Z}$	the set of all integers
Z	the (forward) shift operator on $H^2(\rho)$, in 3.20 – 3.22; in 3.24, a (forward) shift operator on $A^2(\mathbf{C})$
$\beta\mathbf{N}$	the Stone-Čech compactification of $\mathbf{N}$
γ_N	the (Lebesgue) volume of the unit ball in $\mathbf{C}^N$
$\Gamma(x)$	Euler's gamma-function, $n! = \Gamma(n+1)$
Δ	the Laplace operator
Δ_h	the Laplace-Beltrami operator on $\mathbf{D}$, see 5.7
$d\eta$	an invariant measure on $\mathbf{D}$, $d\eta(z) = (1 - \lvert z\rvert^2)^2 \, d\nu(z)$
ϕ_k	the k-th Fourier coefficient of $\phi \in L^2(\mathbf{T})$
Ω	a domain in $\mathbf{C}^N$, or $\mathbf{C}^N$ (see the Convention at the beginning of Chapter 1)
$\lvert\Omega\rvert$	the Lebesgue measure of Ω
$\overline{\Omega}, (\partial\Omega)$	the closure (boundary) of $\Omega \subset \mathbf{C}^N$ in $\mathbf{G}^N$
π	the canonical projection of $\mathcal{B}(H)$ onto $\texttt{Calk}(H)$
$\pi_n(\mathbf{T}), \pi(\mathcal{S}, \mathbf{T})$	homotopy groups
σ	the spectrum
σ_e	the essential spectrum
τ_B, τ_H, τ_F	the set of all Toeplitz operators with bounded symbol, on $A^2(\mathbf{D})$, H^2, $A^2(\mathbf{C}^N)$, respectively
ξ, ξ_B, ξ_F	symbol maps on H^2, $A^2(\mathbf{D})$, $A^2(\mathbf{C}^N)$, see page 46 & ff.

If $n = (n_1, n_2, \ldots, n_N)$ is a multiindex, then

$$a_n = a_{n_1,n_2,\ldots,n_N}, \qquad n! = n_1!n_2!\ldots n_N!,$$

$$2^n = 2^{n_1+n_2+\cdots+n_N}, \qquad z^n = z_1^{n_1} z_2^{n_2} \ldots z_N^{n_N}.$$

If $x, y \in \mathbf{C}^N$, then

$$\overline{x}y = y\overline{x} = \sum_{j=1}^{N} \overline{x}_j y_j, \qquad |x| = (\overline{x}x)^{1/2}.$$

If $\omega = x + yi \in \mathbf{C}$, then

$$\frac{\partial}{\partial\omega} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial\overline{\omega}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

$:=$	equals by definition
$[A, B]$	$= AB - BA$, the commutator of A and B
$[T_f, T_g)$	$= T_{fg} - T_f T_g$, the semicommutator
$T^\sharp$	an operator on $\mathcal{L}$, see before 4.4
$'$	the first derivative, or the commutant (exception: $\mathcal{L}'$)
$''$	the double commutant (exception: $\mathcal{L}''$)
$\widehat{T}$	see before 4.19
$\sim$	the Berezin transform, on $L^\infty(\mathbf{D})$, $L^\infty(\mathbf{C}^N)$, $\mathcal{B}(A^2(\mathbf{D}))$, $\mathcal{B}(A^2(\mathbf{C}^N))$
$\sim^{(k)}$	the k-tuple iterate of $^\sim$
$\flat$	a smoothing transformation, see before 1.11
$\mathbf{1}$	the constant function equal to one
$\\|\,.\,\\|_B$	norm on a Banach space B; B is frequently omitted
$\langle\,.\,,\,.\,\rangle_H$	scalar product on a Hilbert space H; H is sometimes omitted
$\\|\,.\,\\|_p$	norm on L^p
$\\|\,.\,\\|_\rho$, $\langle\,.\,,\,.\,\rangle_\rho$	norm & scalar product on $H^2(\rho)$
$\\|\,.\,\\|_{\mathcal{L}}$, $\langle\,.\,,\,.\,\rangle_{\mathcal{L}}$	norm & scalar product on $\mathcal{L}$
$\\|\,.\,\\|_X$	norm in X_1
$\\|\,.\,\\|_e$	the essential norm (of an operator)
$\\|\,.\,\\|_{p\to p}$, $\\|\,.\,\\|_{\mathcal{L}\to\mathcal{L}}$	norm of an operator from L^p into L^p, from $\mathcal{L}$ into $\mathcal{L}$
$\to$	converges
$\rightrightarrows$	converges uniformly
$\nearrow$	converges increasingly
$\xrightarrow{\text{w}}$	converges weakly (on a Banach space)
$*$	convolution
$\upharpoonright$	restriction

CONTENTS

OBSAH

ADDENDUM

After finishing this thesis, the author has received a preprint of Berger and Coburn [BC], and also a new article of Zhu [Z] has appeared. In [BC], our Theorem 2.4 is proved for the special case $\Omega = \mathbf{C}^N$ (Theorem 9); the proof makes heavy use of the machinery developed by Berezin, but probably could be reduced to the method employed in our "first proof" of Theorem 2.4. On page 38, they conjecture that "The C^*-algebra generated by all $\{T_g : g$ bounded $\}$ is evidently very large. Despite Theorem 16, this algebra could contain all bounded operators." A negative answer to this conjecture is provided by our results in Chapter 3 (Theorem 3.27). In [Z], 5.6 and 5.7 appear, and 5.14 is reported to have been proved in the special case $p = 2$.

[BC] C.A. Berger, L.A. Coburn: Berezin-Toeplitz estimates. Preprint, ? October 1990.

[Z] K.H. Zhu: On certain unitary operators and composition operators. Proc. Symp. Pure Math. 51, 371 – 385. Providence, 1990.